In The Name of Allah

Digital Media Laboratory
Sharif University of Technology

Statistical Pattern Recognition

Graphical Methods

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http://ce.sharif.edu/courses/90-91/2/ce725-1/

Agenda

- **Graphical Methods**
- **Bayesian Networks**
  - Constructing Bayesian Network
  - Inference (General Querying)
  - Stochastic Simulation
  - Learning of Bayesian Networks
- **Hidden Markov Models (HMM)**
  - Markov Random Processes
  - Motivation and Background
  - Specification of an HMM
  - Central problems in HMM modelling
  - Word Recognition Example
  - Character recognition with HMM example

Graphical Methods

- **The use of probabilistic (specifically Bayesian) methods in pattern recognition amounts to a consistent application of the sum and product rules of probability**
- **These lead to complicated algebraic formulae. Hence, it is highly advantageous to augment the analysis using diagrammatic representations of probability distributions, called *probabilistic graphical models*.**

Graphical Methods

- **Key Idea:**
  - Conditional independence assumptions useful.
  - Graphical models express sets of conditional independence assumptions via graph structure
  - Graph structure plus associated parameters define joint probability distribution over set of variables/nodes

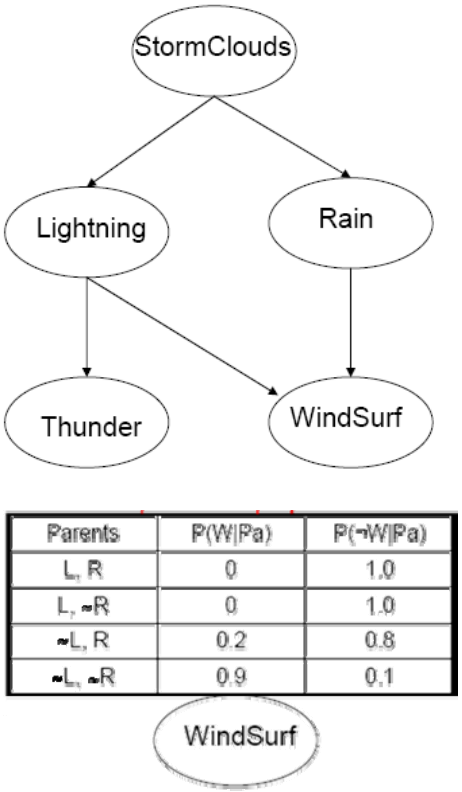
Graphical Methods

- **Advantages**
  - They provide a simple way to visualize the structure of a probabilistic model and can be used to design and motivate new models.
  - Insights into the properties of the model can be obtained by inspection of the graph.
  - Complex computations, required to perform inference and learning in sophisticated models, can be expressed in terms of graphical manipulations, in which underlying mathematical expressions are carried along implicitly.
- **Two types of graphical models:**
  - Undirected graphs (e.g. Markov Random Fields)
  - Directed graphs (e.g. Bayesian Networks and HMM)

Bayesian Networks

- **Bayesian network**
  - **Also called a belief network**
  - **a directed acyclic graph defining a joint probability distribution over a set of variables**
  - **Node denote random variables and edges denote dependencies**
  - ***Each node is conditionally independent of its non-descendents, given its immediate parents.***
  - **A conditional probability distribution (CPD) is associated with each node N, defining P(N | Parents(N))**
  - **In Bayesian net we have:**

$$P(X_1,\ldots,X_n) = \prod_i P(X_i \mid Parent(X_i))$$



Constructing Bayesian Network

- **Algorithm**
  - **Choose an ordering over variables, e.g., X_1, X_2, ..., X_n**
  - **For i=1 to n**
    - Add X_i to the network
    - Select parents Parent(X_i) as minimal subset of X_1, ..., X_{i-1} such that

$$P(X_i \mid Parent(X_i)) = P(X_i \mid X_1,...,X_{i-1})$$

    - Notice this choice of parents assures

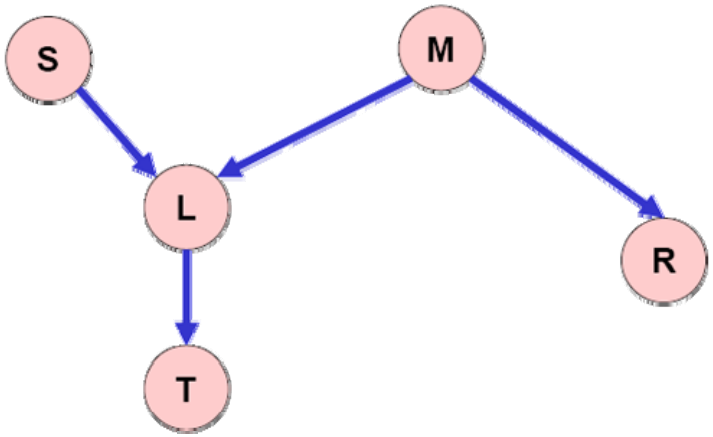
$$P(X_1,...,X_n) = \prod_i P(X_i \mid X_1, X_2,...,X_{i-1}) = \prod_i P(X_i \mid Parent(X_i))$$

Example

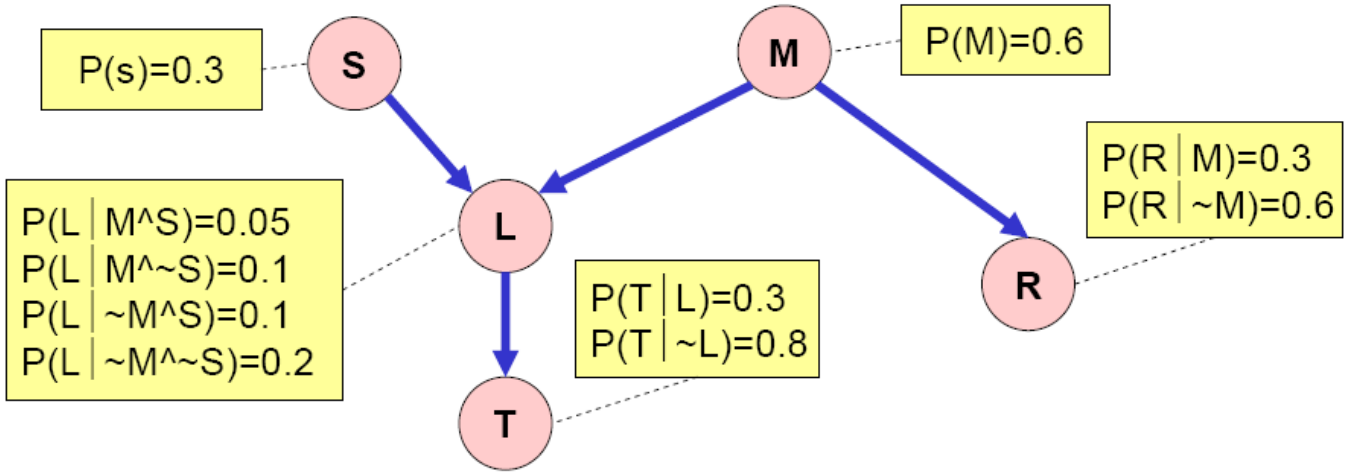
- **Assume five variables**
  - **T: The lecture started by 10:35**
  - **L: The lecturer arrives late**
  - **R: The lecture concerns robots**
  - **M: The lecturer is Manuela**
  - **S: It is sunny**
- **Assumptions**
  - **T only directly influenced by L (i.e. T is conditionally independent of R,M,S given L)**
  - **L only directly influenced by M and S (i.e. L is conditionally independent of R given M & S)**
  - **R only directly influenced by M (i.e. R is conditionally independent of L,S, given M)**
  - **M and S are independent**

Example

- **Step 1: Add variables**
  - Just choose the variables you'd like to be included in the net
- **Step 2: add links**
  - The link structure must be acyclic.
  - If node X is given parents $Q_1,Q_2,..Q_n$ you are promising that any variable that's a non-descendent of X is conditionally independent of X given $\{Q_1,Q_2,..Q_n\}$
- **Step 3: add a probability table for each node.**
  - The table for node X must list P(X|Parent(x)) for each possible combination of parent values

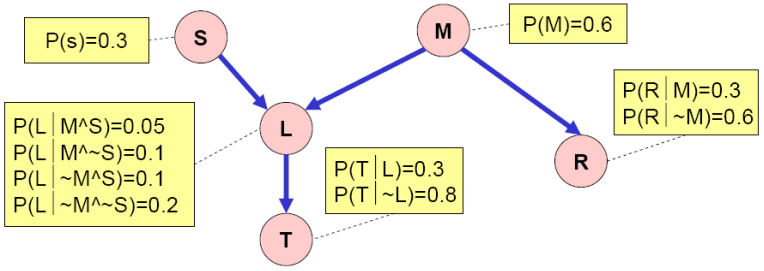


Example



- **Note that:**
  - Two unconnected variables may still be correlated (why?)
  - Each node is conditionally independent of all non-descendants in the tree, given its parents.
  - You can deduce many other conditional independence relations from a Bayesian net.

Example



- **Computing with Bayesian Network (inference)**
  - **P(T ^ ~R ^ L ^ ~M ^ S) = ?**

P(T ^ ~R ^ L ^ ~M ^ S) = P(T | ~R ^ L ^ ~M ^ S) * P(~R ^ L ^ ~M ^ S) =

P(T | L) * P(~R ^ L ^ ~M ^ S) =

P(T | L) * P(~R | L ^ ~M ^ S) * P(L^~M^S) =

P(T | L) * P(~R | ~M) * P(L^~M^S) =

P(T | L) * P(~R | ~M) * P(L|~M^S)*P(~M^S) =

P(T | L) * P(~R | ~M) * P(L|~M^S)*P(~M | S)*P(S) =

P(T | L) * P(~R | ~M) * P(L|~M^S)*P(~M)*P(S).

Inference (General Querying)

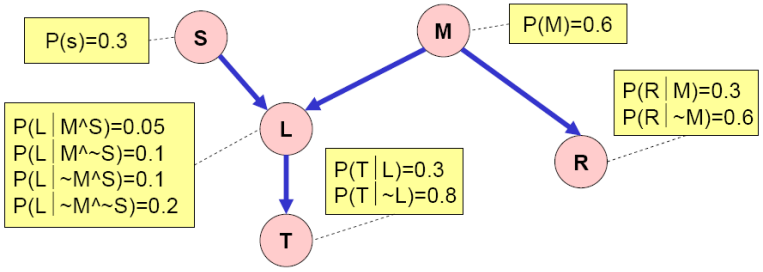
- **We can compute answers to any questions**
  - **E.G. What could we do to compute P(R | T,~S)?**

$$P(R \mid T, \sim S) = \frac{P(R \wedge T \wedge \sim S)}{P(R \wedge T \wedge \sim S) + P(\sim R \wedge T \wedge \sim S)}$$

- **Answering queries need some tricks**
  - **There is a general querying of Bayesian nets, but it is NP-complete, unfortunately.**
  - **A popular, practical approach is Stochastic Simulation.**

Stochastic Simulation

- **It's pretty easy to generate a set of variable-assignments at random with the same probability as the underlying joint distribution.**
- **How?**



1. Randomly choose S. S = True with prob 0.3
2. Randomly choose M. M = True with prob 0.6
3. Randomly choose L. The probability that L is true depends on the assignments of S and M. E.G. if steps 1 and 2 had produced S=True, M=False, then probability that L is true is 0.1
4. Randomly choose R. Probability depends on M.
5. Randomly choose T. Probability depends on L

Stochastic Simulation Algorithm

- **Let's generalize the example on the previous slide to a general Bayesian net**
  - **Call the variables X_1, ..., X_n, where Parent(X_i) must be a subset of {X_1, ..., X_{i-1}}.**
  - **For i = 1 to n:**
    1. Find parents, if any, of X_i. Assume n(i) parents. Call them $X_{p(i,1)}$, $X_{p(i,2)}$, …, $X_{p(i,n(i))}$.
    2. Recall the values that those parents were randomly given: $x_{p(i,1)}$, $x_{p(i,2)}$, …, $x_{p(i,n(i))}$.
    3. Look up in the lookup-table for

$$P(X_i=\text{True} \mid X_{p(i,1)}=x_{p(i,1)}, X_{p(i,2)}=x_{p(i,2)}, \ldots, X_{p(i,n(i))}=x_{p(i,n(i))})$$

    4. Randomly set x_i=True according to this probability
  - **x_1, x_2, …, x_n are now a sample from the joint distribution of X_1, X_2, …, X_n.**

Stochastic Simulation Example

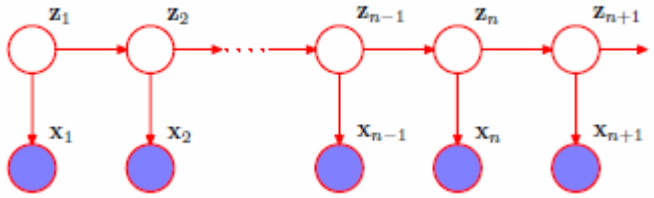
- **Someone wants to know P(R = True | T = True ^ S = False)**
- **We'll do lots of random samplings and count the number of occurrences of the following:**
  - **N_c: Number of samples in which T=True and S=False.**
  - **N_s: Number of samples in which R=True, T=True and S=False.**
  - **N: Number of random samplings**
- **Now if N is big enough:**
  - **N_c/N is a good estimate of P(T=True and S=False).**
  - **N_s/N is a good estimate of P(R=True ,T=True , S=False).**
  - **P(R|T^~S) = P(R^T^~S)/P(T^~S), so N_s/N_c can be a good estimate of P(R|T^~S).**

Learning of Bayesian Networks

- **Variants of learning task:**
  - **Network structure**
    - Structure might be unknown
  - **Networks variables**
    - Training examples might provide values of only some variables
    - If structure of network be known we can use for example gradient ascent algorithm for training the variables
- **We don't cover the learning tasks in this course!**

Hidden Markov Models (HMMs)

- **A specific, and very familiar, example of a directed graph**
  - **It is widely used in speech recognition, handwriting recognition, DNA analysis, and other sequential data applications**



Markov Random Processes

- **Markov random process**
  - A random sequence has the Markov property if its distribution is determined solely by its current state. Any random process having this property is called a *Markov random process*.
- **Markov chain**
  - For observable state sequences (state is known from data), this leads to a *Markov chain* model.
- **Hidden Markov Model**
  - For non-observable states, this leads to a *Hidden Markov Model* (HMM).

HMM Motivation

- **Real-world has structures and processes which have (or produce) observable outputs**
  - **Usually sequential (process unfolds over time)**
  - **Cannot see the event producing the output**
    - Example: speech signals
    - Example: Weather forecasting
- **Problem: how to construct a model of the structure or process given only observations?**

HMM Background

- **Basic theory developed and published in 1960s and 70s**
- **No widespread understanding and application until late 80s**
- **Why?**
  - **Theory published in mathematic journals which were not widely read by practicing engineers**
  - **Insufficient tutorial material for readers to understand and apply concepts**

HMM Uses

- **Some uses of HMM are:**
  - **Speech recognition and processing**
    - Recognizing spoken words and phrases
    - Speech synthesis
    - Many applications in this field
  - **Text processing**
    - Parsing raw records into structured records
    - Part-of-Speech Tagging
  - **Bioinformatics**
    - Protein sequence prediction
  - **Financial**
    - Stock market forecasts (price pattern prediction)
    - Comparison shopping services

Specification of an HMM

- **Components**
  - **N - number of states**
    - $Q = \{q_1; q_2; : : : ;q_T\}$ - set of states
  - **M - number of symbols (observables)**
    - $O = \{o_1; o_2; : : : ;o_T\}$ - set of symbols
  - **A - the state transition probability matrix**
    - $a_{ij} = P(q_{t+1} = j|q_t = i)$
  - **B- observation probability distribution**
    - $b_j(k) = P(o_t = k|q_t = j) \quad i \le k \le M$
  - **π - the initial state distribution**
- **Full HMM is thus specified as a triplet:**
  - **λ = (A,B,π)**

Central problems in HMM modelling

- **Problem 1: Evaluation:**
  - Probability of occurrence of a particular observation sequence, $O = \{o_1,\ldots,o_k\}$, given the model - $P(O|\lambda)$
  - Complicated – hidden states
  - Useful in sequence classification
- **Problem 2: Decoding:**
  - Optimal state sequence to produce given observations, $O = \{o_1,\ldots,o_k\}$, given model
  - Optimality criterion
  - Useful in recognition problems
- **Problem 3: Learning:**
  - Determine optimum model, given a training set of observations
  - Find λ, such that $P(O|\lambda)$ is maximal

Problem 1: Naïve solution

- **State sequence $Q = (q_1,\ldots q_T)$**
- **Assume independent observations:**

$$P(O \mid q,\lambda) = \prod_{i=1}^{T} P(o_t \mid q_t,\lambda) = b_{q1}(o_1)b_{q2}(o_2)\ldots b_{qT}(o_T)$$

  - **Observations are mutually independent, given the hidden states. (Joint distribution of independent variables factorises into marginal distributions of the independent variables.)**

- **Observe that:** $P(q \mid \lambda) = \pi_{q_1} a_{q_1 q_2} a_{q_2 q_3} \ldots a_{q_{T-1} q_T}$
- **And that:** $P(O \mid \lambda) = \sum_{q} P(O \mid q,\lambda) P(q \mid \lambda)$

Problem 1: Efficient solution

- **Forward algorithm:**
  - **A Dynamic Programming approach**
  - **Define auxiliary forward variable α:** $\alpha_t(i) = P(o_1,...,o_t \mid q_t = i,\lambda)$
    - $\alpha_t(i)$ is the probability of observing a partial sequence of observables $o_1,\ldots o_t$ such that at time t, state q_t=i
  - **Recursive algorithm:**
    - Initialise: $\alpha_1(i) = \pi_i b_i(o_1)$
    - Calculate: $\alpha_{t+1}(j) = [\sum_{i=1}^{N} \alpha_t(i) a_{ij}]\, b_j(o_{t+1})$
    - Obtain: $P(O \mid \lambda) = \sum_{i=1}^{N} \alpha_T(i)$

Problem 1: Alternative solution

- **Backward algorithm:**
  - **Again Dynamic Programming**
  - **Define auxiliary forward variable *β*:** $\beta_t(i) = P(o_{t+1}, o_{t+2}, \ldots, o_T \mid q_t = i, \lambda)$
    - $\beta_t(i)$: the probability of observing a sequence of observables $o_{t+1}, \ldots, o_T$ given state q_t =i at time t, and λ
  - **Recursive algorithm:**
    - Initialise: $\beta_T(j) = 1$
    - Calculate: $\beta_t(i) = \sum_{j=1}^{N} \beta_{t+1}(j) a_{ij} b_j(o_{t+1})$
    - Terminate: $p(O \mid \lambda) = \sum_{i=1}^{N} \beta_1(i)$

Problem 2: Decoding

- Choose state sequence to maximise probability of observation sequence
- Viterbi algorithm - inductive algorithm that keeps the best state sequence at each instance
  - Utilizes dynamic programming
  - State sequence to maximise: $P(q_1, q_2, ...q_T | O, \lambda)$
  - Define auxiliary variable δ: $\delta_t(i) = \max_q P(q_1, q_2, ..., q_t = i, o_1, o_2, ...o_t | \lambda)$
    - $\delta_t(i)$ – the probability of the most probable path ending in state $q_t=i$

Problem 2: Decoding

- **Recurrent property:** $\delta_{t+1}(j) = \max_i(\delta_t(i)a_{ij})b_j(o_{t+1})$
- **Algorithm:**
  - To get state seq, need to keep track of argument to maximise this, for each *t* and *j*. Done via the array $\psi t(j)$.
  - **1. Initialise:** $\delta_1(i) = \pi_i b_i(o_1), \quad 1 \le i \le N$
    $\psi_1(i) = 0$
  - **2. Recursion:** $\delta_t(j) = \max_{1\le i\le N}(\delta_{t-1}(i)a_{ij})b_j(o_t)$
    $\psi_t(j) = \arg\max_{1\le i\le N}(\delta_{t-1}(i)a_{ij})$
    $2 \le t \le T, 1 \le j \le N$
  - **3. Terminate:** $P^* = \max_{1\le i\le N}\delta_T(i)$
    $q_T^* = \arg\max_{1\le i\le N}\delta_T(i)$
    - **P* gives the state-optimised probability**
    - **Q* is the optimal state sequence (Q* = {q1*,q2*,…,qT*})**
  - **4. Backtrack state sequence:** $q_t^* = \psi_{t+1}(q_{t+1}^*), \quad t+T-1, T-2, \ldots, 1$

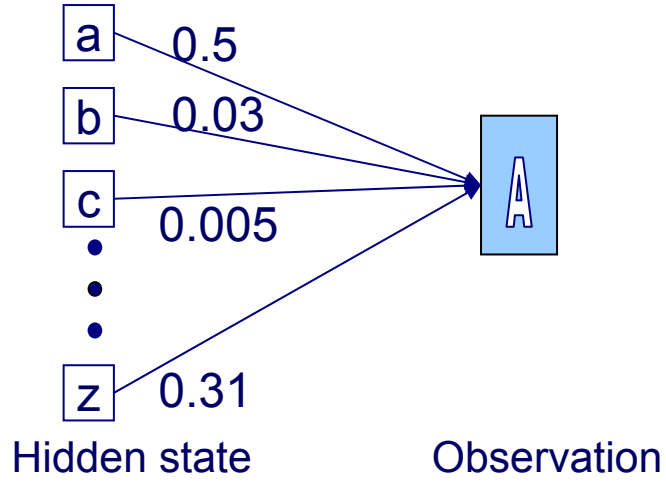
Problem 3: Learning

- **Training HMM to encode observation sequence such that HMM should identify a similar observation sequence in future**
- **Find $\lambda=(A,B,\pi)$, maximising $P(O|\lambda)$**
- **General algorithm:**
  - **Initialise: λ_0**
  - **Compute new model λ, using λ_0 and observed sequence O**
  - **Then $\lambda_o \leftarrow \lambda$**
  - **Repeat steps 2 and 3 until: $\log P(O|\lambda) - \log P(O|\lambda_0) < d$**
- **We don't cover the learning algorithms in this course.**

Word Recognition Example

- **Typed word recognition, assume all characters are separated.**

- **Character recognizer outputs probability of the image being particular character, P (image |character).**

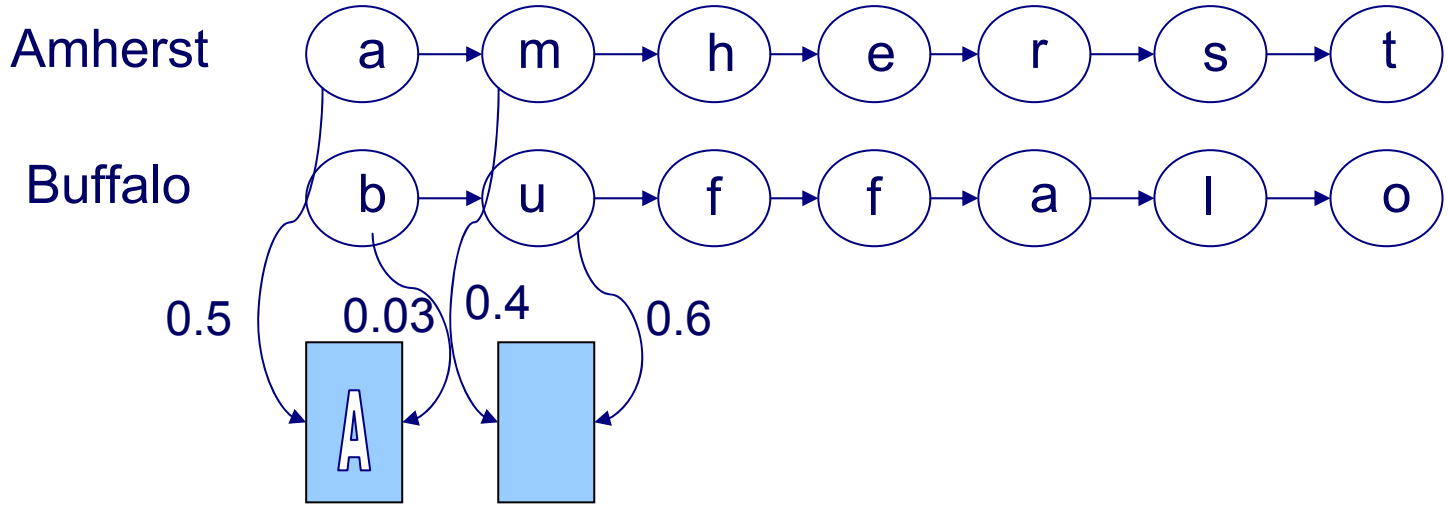


Word Recognition Example

- **Hidden states of HMM = characters.**
- **Observations = typed images of characters segmented from the image v_α. Note that there is an infinite number of observations.**
- **Observation probabilities = character recognizer scores.**
- **Transition probabilities will be defined differently in two subsequent models.**

Word Recognition Example

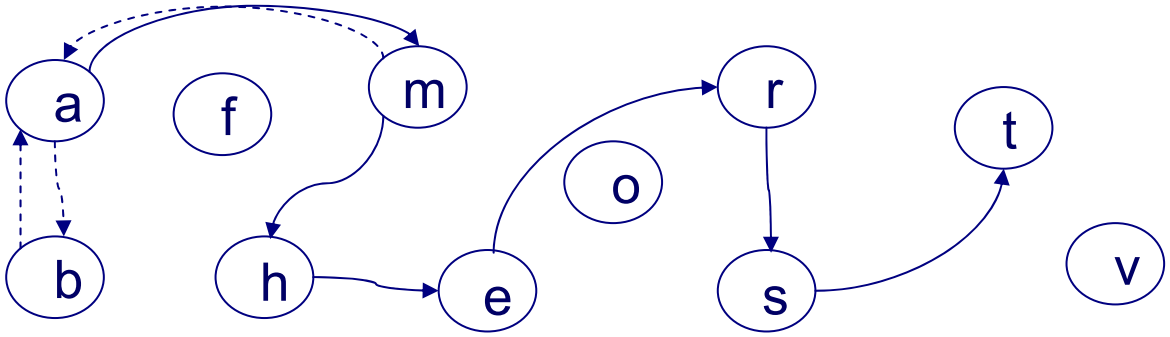
- If lexicon is given, we can construct separate HMM models for each lexicon word.



- Here recognition of word image is equivalent to the problem of evaluating few HMM models.
- This is an application of Evaluation problem.

Word Recognition Example

- We can construct a single HMM for all words.
- Hidden states = all characters in the alphabet.
- Transition probabilities and initial probabilities are calculated from language model.
- Observations and observation probabilities are as before.



- Here we have to determine the best sequence of hidden states, the one that most likely produced word image.
- This is an application of Decoding problem.

Further Reading

- **L. R. Rabiner, "A tutorial on Hidden Markov Models and selected applications in speech recognition," Proceedings of the IEEE, vol. 77, pp. 257-286, 1989.**
- **R. Dugad and U. B. Desai, "A tutorial on Hidden Markov models," Signal Processing and Artifical Neural Networks Laboratory, Dept of Electrical Engineering, Indian Institute of Technology, Bombay Technical Report No.: SPANN-96.1, 1996.**
- **Andrew W. Moore, HMM tutorial, www.autonlab.org/tutorials.**

Any Question?

End of Lecture 12

Thank you!

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http://ce.sharif.edu/courses/90-91/2/ce725-1/

- **Thanks to Andrew W. Moore for his good slides**
  - **www.autonlab.org/tutorials**
- **Thanks to Tom Mitchell for his good slides**
  - **www.cs.cmu.edu/~tom/**