In The Name of Allah

Digital Media Laboratory
Sharif University of Technology

Statistical Pattern Recognition

Classification –Discriminant Functions

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Agenda

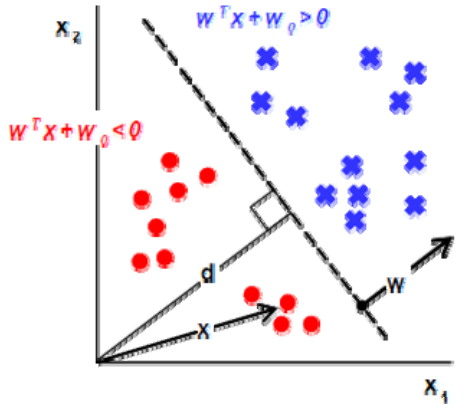
- Linear Discriminant Functions (LDF)
- Multi-class problems
  - Linear machine
  - Completely Linearly Separation
  - Pairwise Linearly Separation
- Linear Discriminant Function Design
  - Least Mean Squared Error Method
  - Sum of Squared Error Method
  - Ho-Kashyap Method
  - Probabilistic Methods

Linear Discriminant Functions (LDF)

- **Definition:**
  - **LDF is a function that is a linear combination of the components of x**

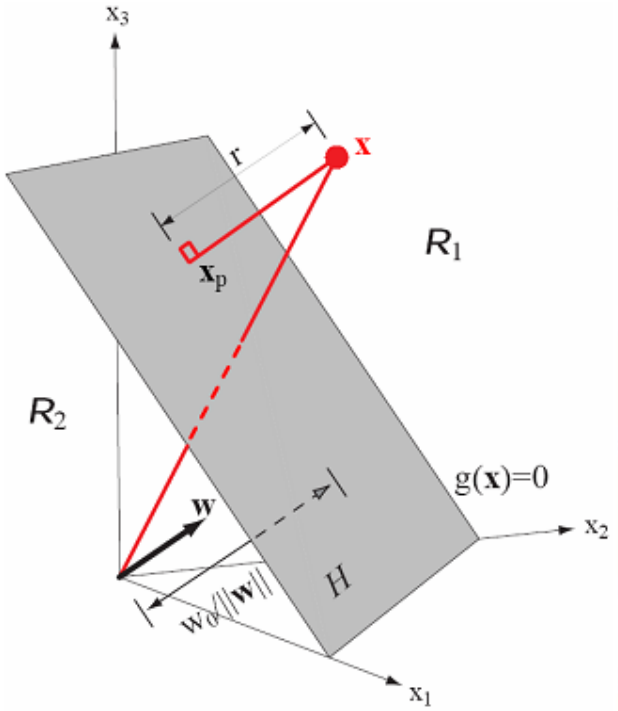
$$g(x) = w^t x + w_0$$

  - **where w is the weight vector and w_0 the bias, or threshold weight.**
- **A two-category classifier with a discriminant function of the above form uses the following rule:**
  - **Decide w_1 if g(x) > 0 and w_2 if g(x) < 0**
    - Decide w_1 if $w^t x > -w_0$ and w_2 otherwise
    - The value g(x) of the function for a certain point **x** is called functional margin
  - **If g(x) = 0 then x is assigned to either class**
    - The equation g(x) = 0 defines the decision surface that separates points assigned to the category w_1 from points assigned to the category w_2
    - When g(x) is linear, the decision surface is a hyperplane.



Linear Discriminant Functions

- In conclusion, a linear discriminant function divides the feature space by a hyperplane decision surface
- Decision boundary g(x)=0 corresponds to (d-1)-dimensional hyperplane in d-dimensional x-space
- The orientation of the surface is determined by the normal vector w and the location of the surface is determined by the bias
  - We can see Fisher method (LDA) as a linear discriminant function, too.



Multi-class problems

- **Suppose we have an n-classes classification problem, and we want to separate them with linear discriminant functions**
  - **Do you have any idea about how to use discriminant function in this case**
    - We have many ways to do this.
- **Using linear discriminant function in multi-class problems**
  - **Linear machines (one versus one)**
  - **Completely linearly separation (one versus the rest)**
  - **Pairwise linearly separation**
- **We introduce above methods through illustrative examples in next slides.**

Case 1: Linear Machine

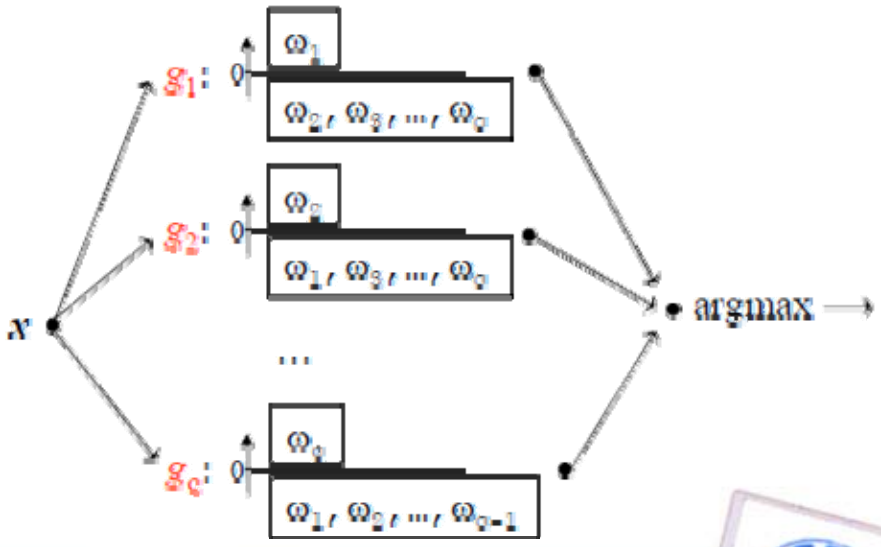
✧ **Suppose a 3-class classification problem with the following discriminant functions:**

$$g_1(x) = -x_1 + x_2$$
$$g_2(x) = x_1 + x_2 - 1$$
$$g_3(x) = -x_2$$

and use the following rule for classification (linear machine rule):

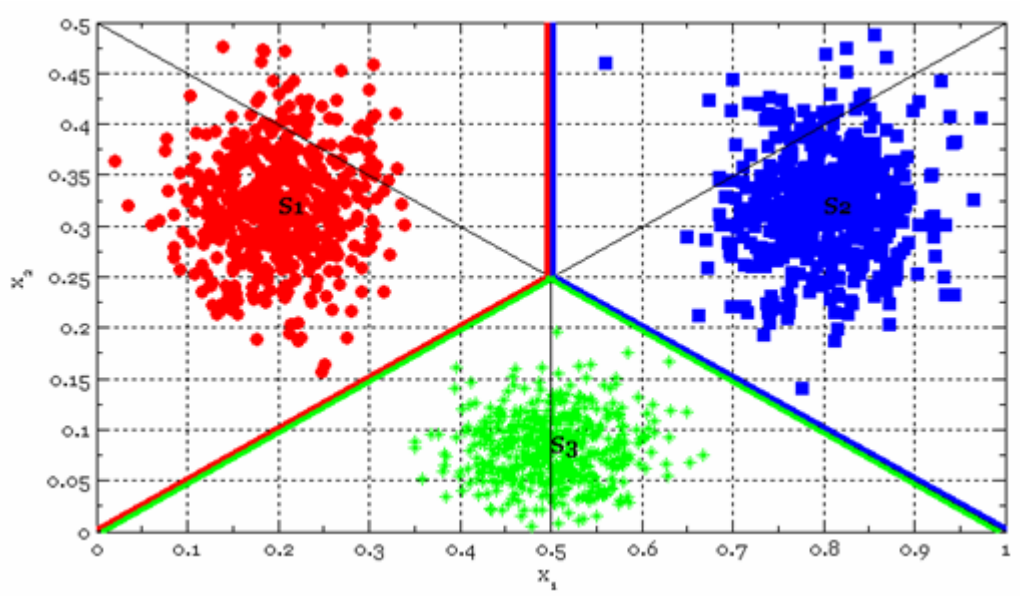
$$x \in C_i \Leftrightarrow g_i(x) < g_j(x) \;\; ; \forall j \neq i$$

How these classes partition the space?



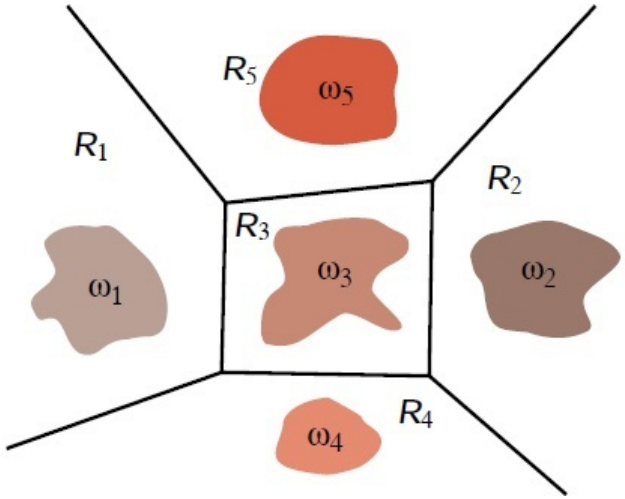
Case 1: Linear Machine

- **Each class partition can be obtained through solving two equations.**
- **The result:**



More on Linear Machines

- In some texts, it is called one versus one (one against one).
- How many functions we need for n classes? (n)



- The decision regions for linear machine are convex and this restriction limits the flexibility of the classifier.

Case 2: Completely Linearly Separation

- **Suppose a 3-class classification problem with the following discriminant functions:**

$$g_1(x) = -x_1 + x_2$$
$$g_2(x) = x_1 + x_2 - 5$$
$$g_3(x) = -x_2 + 1$$

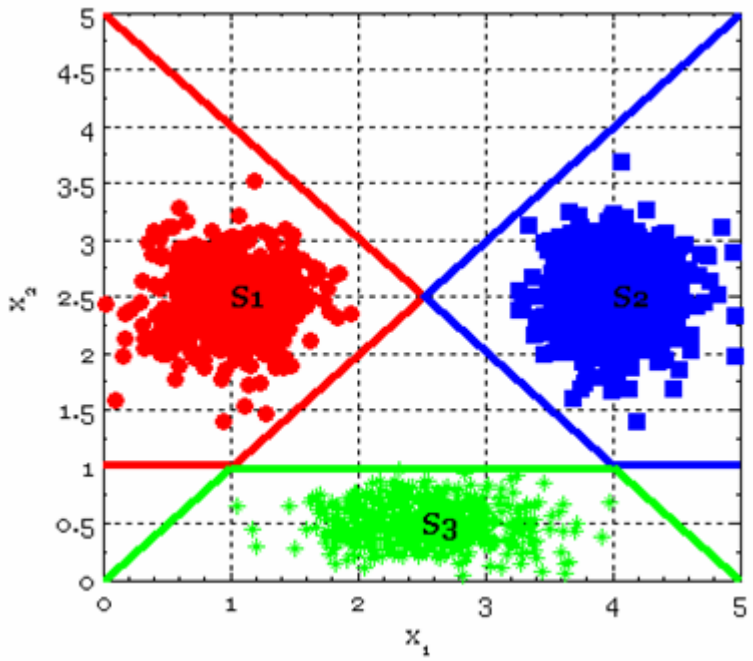
and use the following rule for classification (completely linearly separation rule):

$$if\ g_i(x) > o \Rightarrow x \in C_i \quad and \quad if\ g_i(x) < o \Rightarrow x \notin C_i$$

How these classes partition the space? Determine the undecided sub-spaces.

Case 2: Completely Linearly Separation

- **Each class partition can be obtained through solving three equation.**
- **The result:**



More on Completely Linearly Separation

- **In some texts, it is called one versus the rest (one against all).**
- **If we have n classes, what is the number of needed functions? (n)**
- **Are the decision regions convex?**
- **Compare the undecided sub-spaces in two cases.**

Case 3: Pairwise Linearly Separation

- **Suppose a 3-class classification problem with the following discriminant functions:**

$$g_{12}(x) = -x_1 - x_2 + 5$$
$$g_{13}(x) = -x_1 + 3$$
$$g_{23}(x) = -x_1 + x_2 \qquad g_{ij}(x) = -g_{ji}(x)$$

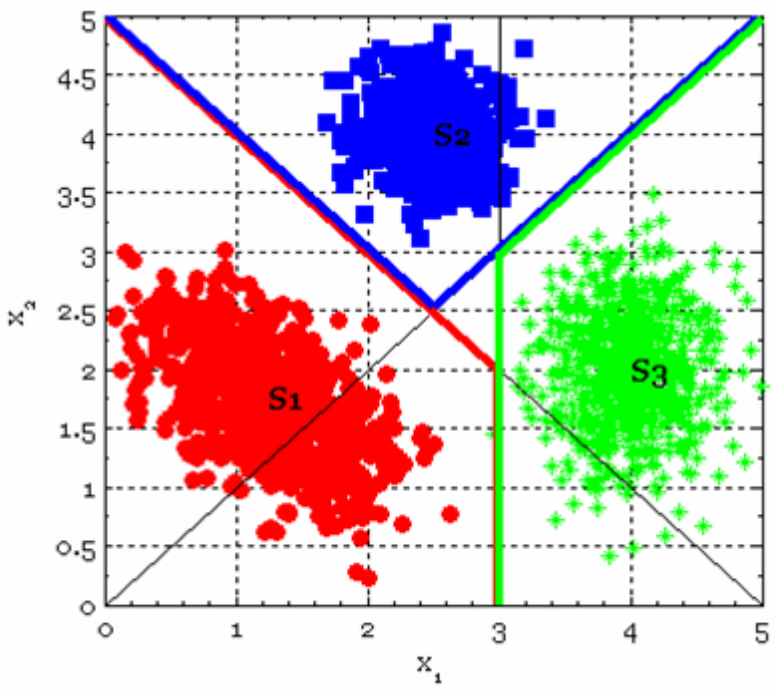
and use the following rule for classification:

$$x \in C_i \Leftrightarrow \forall j \neq i \; g_{ij}(x) > o$$

How these classes partition the space? Determine the undecided sub-spaces.

Case 3: Pairwise Linearly Separation

- **Each class partition can be obtained through solving two equation.**
- **The result:**

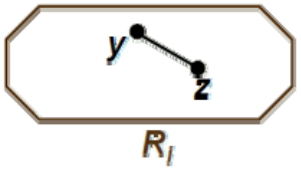


More on Pairwise Linearly Separation

- **If we have n classes, what is the number of needed functions? (c(n,2))**
- **Are the decision regions convex?**

$$\forall j \neq i \quad g_i(y) \geq g_j(y) \text{ and } g_i(z) \geq g_j(z) \Leftrightarrow g_i(\alpha y + (1-\alpha)z) \geq g_j(\alpha y + (1-\alpha)z)$$

- **Definition: A region R_i is convex iff** $\forall y, z \in R_i \Rightarrow \alpha y + (1-\alpha)z \in R_i$



Linear Discriminant Functions

- **Main problem**
  - **How to create the discriminant functions for each class (how obtain w)?**
- **Many methods exist for this purpose, such as:**
  - **Error Minimization Methods**
    - Least Mean Squared Error Method → will be discussed in next slides
    - Sum of Squared Error Method → will be discussed in next slides
    - Ho-Kashyap Method → will be discussed in next slides
  - **Fisher Linear Discriminant Method → discussed in lecture 3**
  - **Perceptorn Method → will be discussed in lecture 9**
  - **Probabilistic Methods → discussed in lecture 6**
  - **etc.**

Least Mean Squared Error

- **We want to choose the W that minimizes the mean-squared-error criterion function:**

$$J(w) = E\left[\left|y - g(x)\right|^2\right] = E\left[(y - w^t x)^2\right]$$

$$\hat{w} = \arg\min_w J(w)$$

$$\frac{\partial J(w)}{\partial w} = 2E\left[x(y - w^t x)\right] = 2E\left[xy - xw^t x)\right] = 2\left(E\left[xy\right] - Ew\left[x^t x\right]\right) = 0$$

$$\hat{w} = \frac{E\left[xy\right]}{E\left[x^t x\right]} = \frac{E\left[xy\right]}{R_x} = R_x^{-1} E\left[xy\right]$$

$$R_x = E[x^t x] = \begin{bmatrix} E[x_1 x_1] & \dots & E[x_1 x_n] \\ E[x_2 x_1] & \cdots & E[x_2 x_n] \\ \vdots & \vdots & \vdots \\ E[x_n x_1] & \cdots & E[x_n x_n] \end{bmatrix}, \quad E[xy] = E\begin{bmatrix} \begin{bmatrix} x_1 y \\ x_2 y \\ \vdots \\ x_n y \end{bmatrix} \end{bmatrix}$$

- **We can also use the gradient descent rule for updating w instead of analytical solving.**

Sum of Squared Error

- **SSE uses the sum of squared error as objective function**
  - **Also known as Pseudo inverse matrix method**

$$J(w)=\left\|w^t x-b\right\|^2=\sum_{i=1}^{n}(w^t x_i-b_i)^2$$

$$\frac{\partial J(w)}{\partial w}=\sum_{i=1}^{n}2x_i(w^t x_i-b_i)=2x(x^t w-b)=0$$

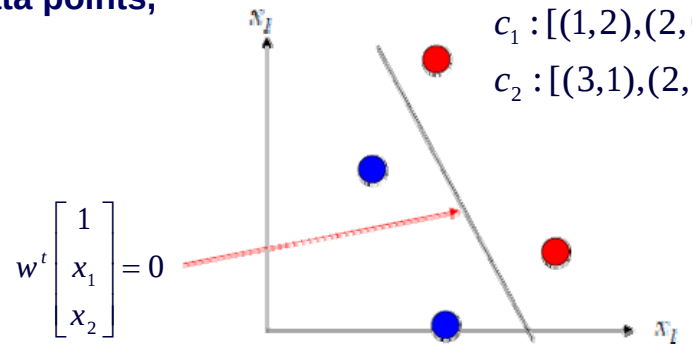
$$xx^t w-xb=0\Rightarrow xx^t w=xb\Rightarrow w=\frac{xb}{xx^t}=\underbrace{(xx^t)^{-1}x}_{x^-}b=x^- b$$

Sum of Squared Error

- **Example**
  - **Find the SSE boundary for the given data points,**

$c_1 : [(1,2),(2,0)]$

$c_2 : [(3,1),(2,3)]$



$$X = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ -1 & -3 & -1 \\ -1 & -2 & -3 \end{bmatrix} \qquad \Rightarrow X^- = (X^t X)^{-1} X^t = \begin{bmatrix} 5/4 & 13/12 & 3/4 & 7/12 \\ -1/2 & -1/6 & -1/2 & -1/6 \\ 0 & -1/3 & 0 & -1/3 \end{bmatrix}$$

$$\text{assuming } y = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \Rightarrow w = X^- y = \begin{bmatrix} 11/3 & -4/3 & -2/3 \end{bmatrix}$$

$$\Rightarrow g(x) = \frac{11}{3} - \frac{4}{3}x_1 - \frac{2}{3}x_2$$

Ho-Kashyap Method

- **The main limitation of the SSE is lack of guarantees that a separating hyperplane will be found in the linearly separable case**
  - **The SSE rule try to minimize** $\|w^t x - b\|^2$
  - **Finding a separating hyperplane depends on how suitably the outputs b are selected**
- **If the two classes are linearly separable, there must exists vectors w and b such that $w^T x = b > 0$**
  - **if b were known, to compute the separating hyperplane, the SSE solution will be $w = x^- b$**
  - **Nevertheless, since b is unknown, one must solve the equation for both w and b**
- **A possible algorithm is the Ho-Kashyap procedure:**
  1. **Find the target values b with gradient descent**
  2. **compute the weight vector w from the SSE solution**
  3. **Repeat 1 and 2 until convergence**

Ho-Kashyap Method

- **g(x) > 0 can be rewrite as g(x)=b; b>0**
  - **How we can determine b?**
- **Objective function in this case is** $J(w,b)=\|w^t x-b\|^2$
- **Ho-Kashyap method offers an iterative method for obtaining w and b, using following steps:**
  - **Keeping constant b and optimize J related to w (using obtained b from last step)**
    - Using previous method we have:

$$w(t+1)=x^{-}b(t)$$

  - **Keeping constant w and optimize J related to b (using obtained w from last step)**
    - The objective is to minimize

$$\frac{\partial J}{\partial w}=-2\left(w^t x-b\right)$$

    - Using Gradient descent method we have:

$$b(t+1)=b(t)+\eta 2\left(w^t(t)x-b\right)$$

    - To hold the constraint b>0, we set (xw-b) in this rule to zero if it becomes negative, then the rule will be:

$$b(t+1)=b(t)+\eta\left[\left(w^t(t)x-b\right)+\left|w^t(t)x-b\right|\right]$$

Probabilistic Methods

- **Maximum likelihood**
  - **$g_i(x) = P(x|w_i)$**
- **Bayesian Classifier**
  - **$g_i(x) = P(w_i|x)$**
    - $g_i(x) = p(x|w_i)\, P(w_i)$
    - $g_i(x) = \ln p(x|w_i) + \ln P(w_i)$
- **Expected Loss (Conditional Risk)**
  - **Uses loss function $\lambda(a_i|w_j)$: is the loss incurred for taking action a_i when the state of nature is w_j.**
  - **$R(a_i|x) = \Sigma_j \lambda(a_i|w_j)\, P(w_j|x)$**
    - We must minimize R for each class.

Any Question

End of Lecture 7

Thank you!

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