In The Name of Allah

Digital Media Laboratory
Sharif University of Technology

Statistical Pattern Recognition

Support Vector Machine (SVM)

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Agenda

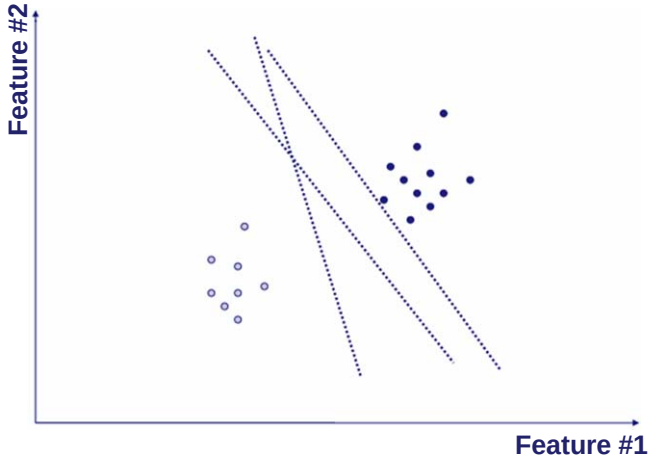
- Introduction
  - Some Concepts
- Linear, Hard-Margin SVM
- Linear, Soft-Margin SVM
- Nonlinear SVM
- SVMs vs. Neural Networks
- Conclusion

Support Vector Machines

- **Decision surface is a hyperplane (line in 2D) in feature space.**
- **SVM three main ideas:**
  - **Define what an optimal hyperplane is: maximize margin**
  - **Extend the above definition for non-linearly separable problems: have a penalty term for misclassifications**
  - **Map data to high dimensional space where it is easier to classify with linear decision surfaces: reformulate problem so that data is mapped implicitly to this space**

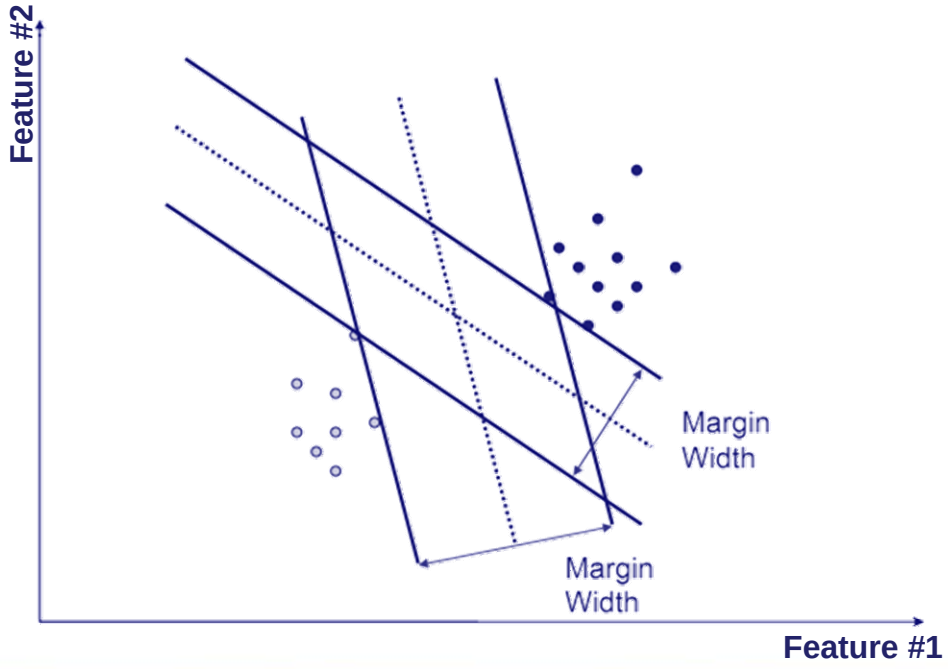
Concepts

- **Which of the hyperplanes should we choose?**
  - **Intuitively, a hyperplane that passes too close to the training examples will be sensitive to noise and, therefore, less likely to generalize well for data outside the training set.**
  - **Instead, it seems reasonable to expect that a hyperplane that is farthest from all training examples will have better generalization capabilities.**



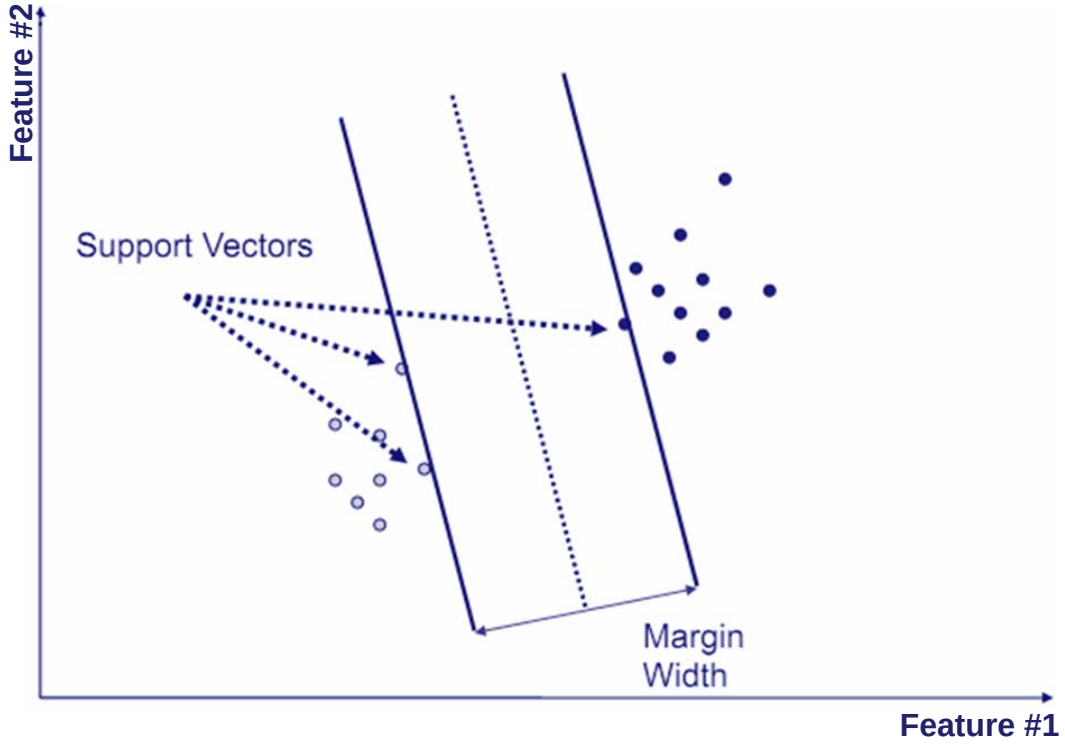
Concepts

- **Maximizing the Margin**
  - **The optimal separating hyperplane will be the one with the largest margin, which is defined as the minimum distance of an example to the decision surface.**



Concepts

- **Support Vectors**



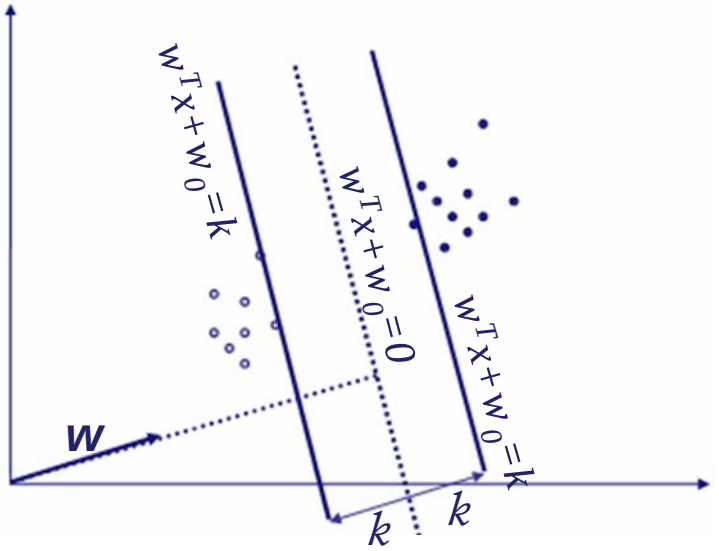
Linear, Hard-Margin SVM

- **The distance of the data point x_0 from the line $G(x)=w^Tx+w_0$ will be** $\frac{G(x_0)}{\|w\|}$
- **The width of the margin is:** $\frac{2|k|}{\|w\|}$
- **So, the problem is:**

$$\max \frac{2|k|}{\|w\|} \quad s.t. \begin{cases} w^T x + w_0 \geq k, \ \forall x \in C_1 \\ w^T x + w_0 \leq -k, \ \forall x \in C_2 \end{cases}$$

- **Canonical form can be used, without loss the generality, then the problem becomes:**

$$\max \frac{2|k|}{\|w\|} \quad s.t. \begin{cases} w^T x + w_0 \geq 1, \ \forall x \in C_1 \\ w^T x + w_0 \leq -1, \ \forall x \in C_2 \end{cases}$$



Linear, Hard-Margin SVM

- **If class 1 corresponds to 1 and class 2 corresponds to -1, we can rewrite**

$$w^T x + w_0 \geq 1, \ \forall x \ with \ y_i = 1$$
$$w^T x + w_0 \leq -1, \ \forall x \ with \ y_i = -1$$

As $y_i(w^T x + w_0) \geq 1, \ \forall x_i$

- **So, the problem becomes**

$$\max \frac{2}{\|w\|} or \min \frac{1}{2}\|w\|^2 \qquad s.t. \ y_i(w^T x_i + w_0) \geq 1, \forall x$$

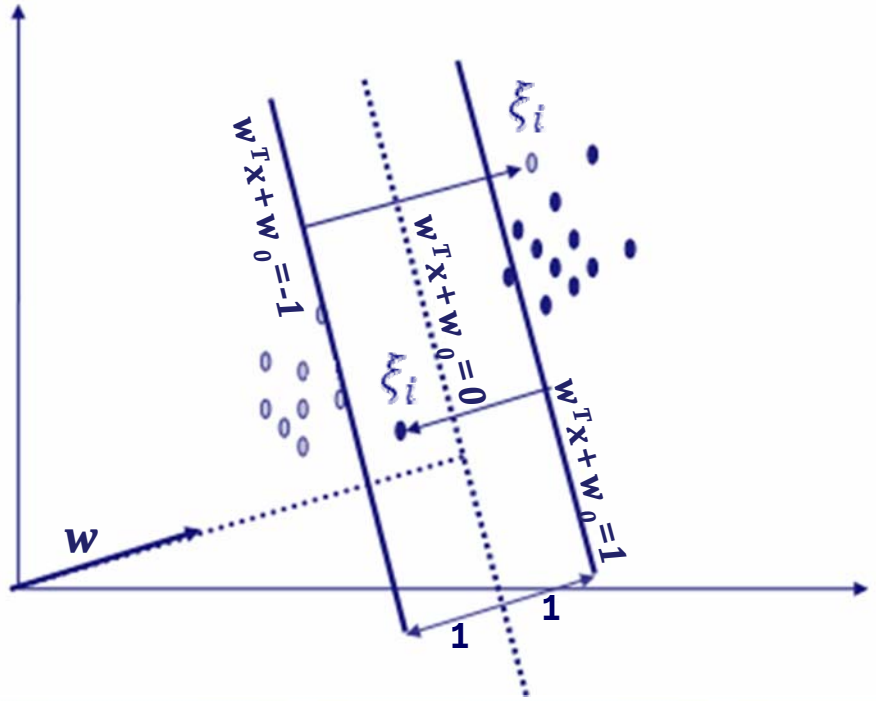
Linear, Hard-Margin SVM

- **Find w, w_0 that solves** $\min \frac{1}{2}\|w\|^2 \; s.t. \; y_i(w^T x_i + w_0) \geq 1, \forall x$
  - This equation will yield the support vectors, too.
- **Problem is convex so, there is a unique global minimum value (when feasible)**
- **There is also a unique minimizer, i.e. (w, w_0) values that provide the minimum**
- **Non-solvable if the data is not linearly separable**
- **Quadratic Programming**
  - Very efficient computationally with modern constraint optimization engines
- **Refer to Farsi notes for more details**

- **Q: How to extend the linear definition for non-linearly separable problems?**

Linear, Soft-Margin SVM

- **Will be used in case of non-linearly separable data**
- **Introduce slack variables (ξ_i), Allow some instances to fall within the margin, but penalize them**

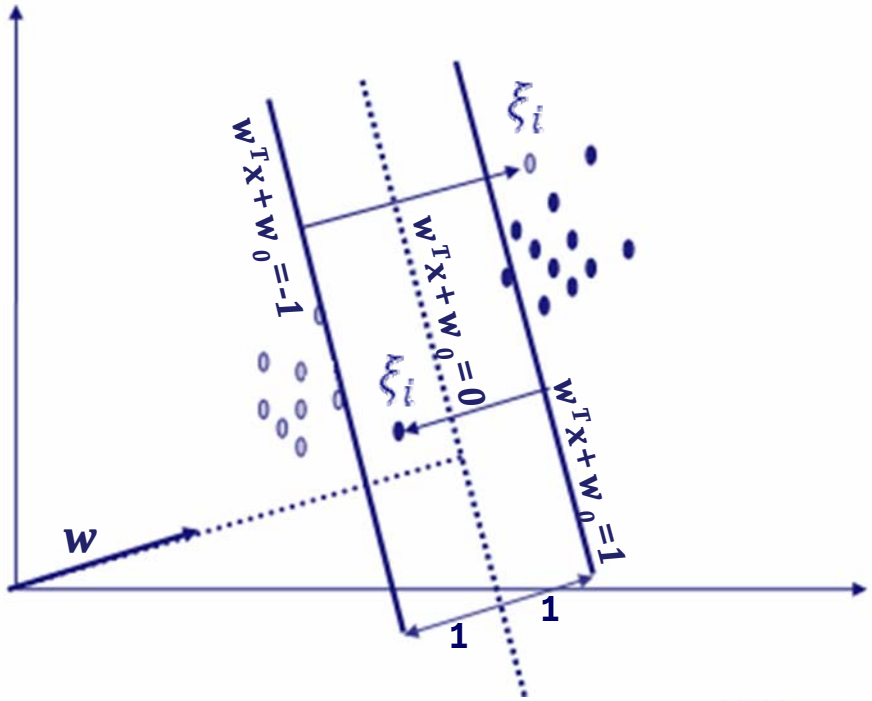


Linear, Soft-Margin SVM

- **Constraint becomes:** $y_i(w^T x_i + w_0) \geq 1 - \xi_i, \forall x_i, \xi_i \geq 0$
- **Objective function penalizes for misclassified instances and those within the margin**

$$\min\left\{\frac{1}{2}\|w\|^2 + c\sum_i \xi_i\right\}$$

  - **C trades-off margin width and misclassifications**



Linear, Soft-Margin SVM

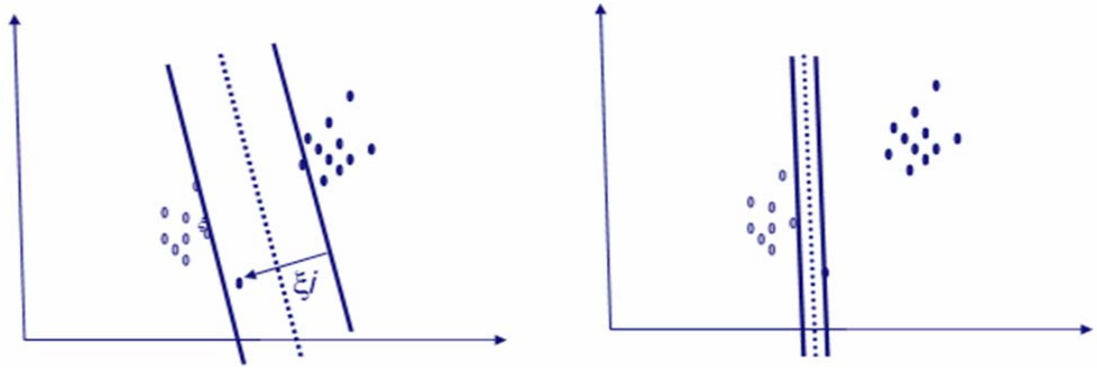
- **Find w, w_0 (and ξ_i s) that solves**

$$\min\left\{\frac{1}{2}\|w\|^2 + c\sum_i \xi_i\right\} \qquad s.t.\ \ y_i(w^T x_i + w_0) \geq 1 - \xi_i, \forall x_i, \xi_i \geq 0$$

  - **SVM tries to maintain ξ_i to zero while maximizing margin.**
  - **Notice: SVM does not minimize the number of misclassifications (NP-complete problem) but the sum of distances from the margin hyperplanes.**
  - **For bigger c values, we get closer to the hard-margin solution.**

Comparison

- ✧ **Soft-Margin always have a solution**
- ✧ **Soft-Margin is more robust to outliers**



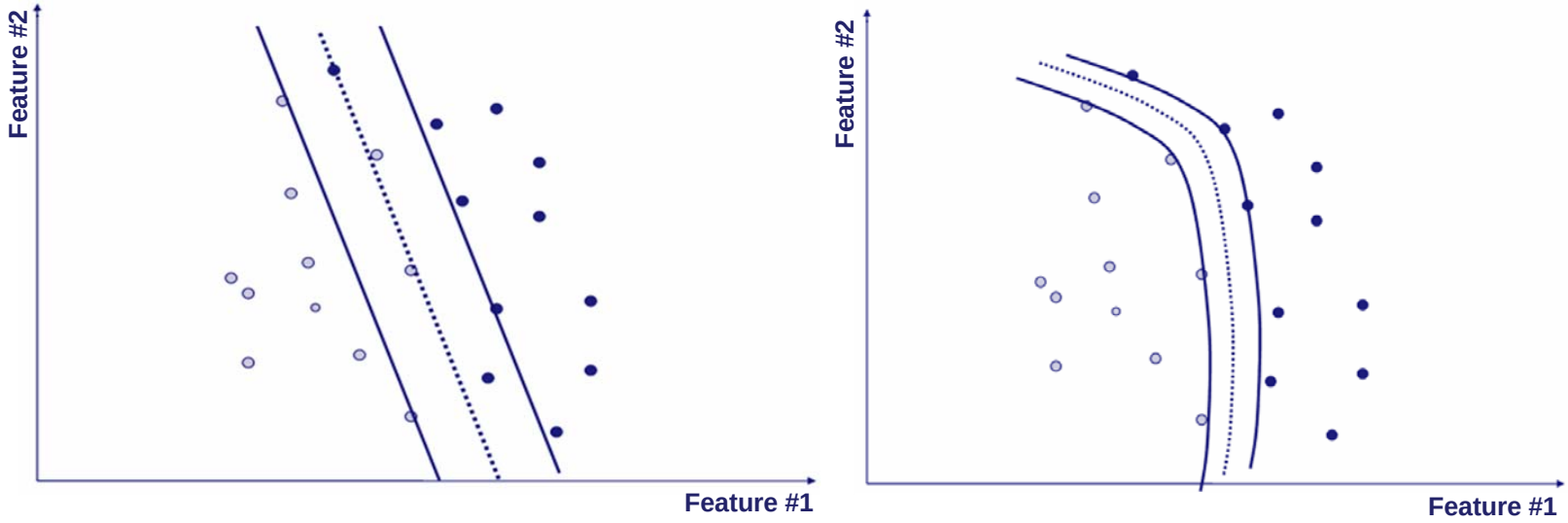
Soft Margin SVM

Hard Margin SVM

- ✧ **Hard-Margin does not require to guess the cost parameter (requires no parameters at all)**

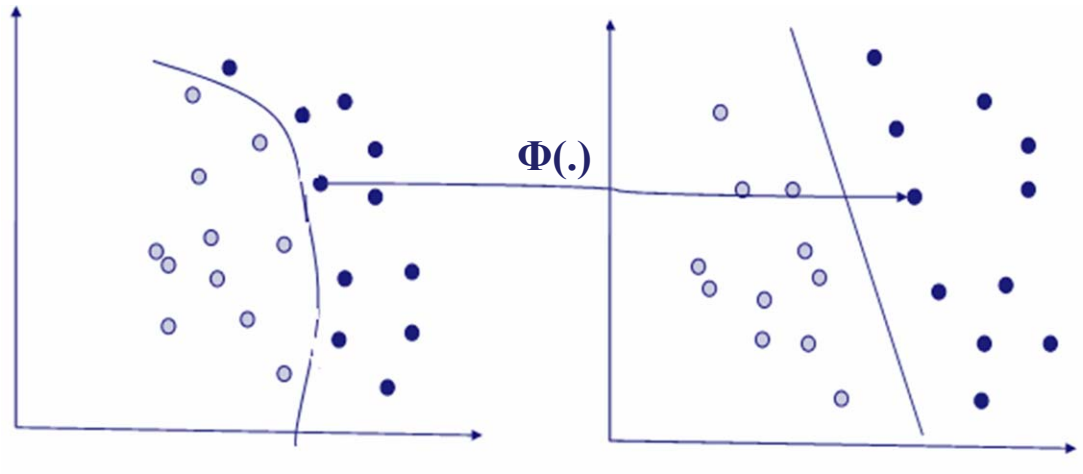
Nonlinear SVM

- **Disadvantages of SVM linear decision surfaces in comparison with nonlinear ones**



Nonlinear SVM

- **Linear classification in high-dimensional spaces**
  - Find function Φ(x) to map to a different space



Nonlinear SVM

- **Using Φ(x) to map to a different space, SVM formulation in this new space becomes**

$$\min\left\{\frac{1}{2}\|w\|^2 + c\sum_i \xi_i\right\} \qquad s.t.\ \ y_i(w^T\phi(x_i) + w_0) \geq 1 - \xi_i, \forall x_i, \xi_i \geq 0$$

  - **Data appear as Φ(x), weights w are now weights in the new space**
  - **Explicit mapping expensive if Φ(x) is very high dimensional**
    - Solving the problem without explicitly mapping the data is desirable
    - Use kernel trick
      - **We'll study "Kernel Methods" in the next session.**
  - **In explicit mapping case, we can solve this optimization problem using its dual form.**

Nonlinear SVM

- **The Dual of the SVM Formulation**
  - **Original SVM formulation**
    - n inequality constraints
    - n positivity constraints
    - n number of x variables

$$\min\left\{\frac{1}{2}\|w\|^2 + c\sum_i \xi_i\right\}$$

$$s.t.\ \ y_i(w^T x_i + w_0) \geq 1 - \xi_i, \forall x_i, \xi_i \geq 0$$

  - **The (Wolfe) dual of this problem**
    - one equality constraint
    - n positivity constraints
    - n number of a variables (Lagrange multipliers)
    - Objective function more complicated

$$\min_{\alpha_i} \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j y_i y_j \phi(x_i)\phi(x_j) - \sum_i \alpha_i$$

$$s.t.\ \ C \geq \alpha_i \geq 0, \sum_i \alpha_i y_i = 0$$

SVMs vs. Neural Networks

SVMs

- Kernel maps to a high dimensional spaces
- Search space has a unique minimum
- Training is extremely efficient
- Classification extremely efficient
- Kernel and cost are the two parameters to select
- Very good accuracy in typical domains
- Extremely robust

Neural Networks

- Hidden Layers map to lower dimensional spaces
- Search space has multiple local minima
- Training is expensive
- Classification extremely efficient
- Requires number of hidden units and layers
- Very good accuracy in typical domains
- Could be robust

Conclusions

- **SVMs express learning as a mathematical program taking advantage of the rich theory in optimization**
- **SVM uses the kernel trick to map indirectly to extremely high dimensional spaces**
- **SVMs extremely successful, robust, efficient, and versatile while there are good theoretical indications as to why they generalize well**

Any Question?

End of Lecture 10

Thank you!

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