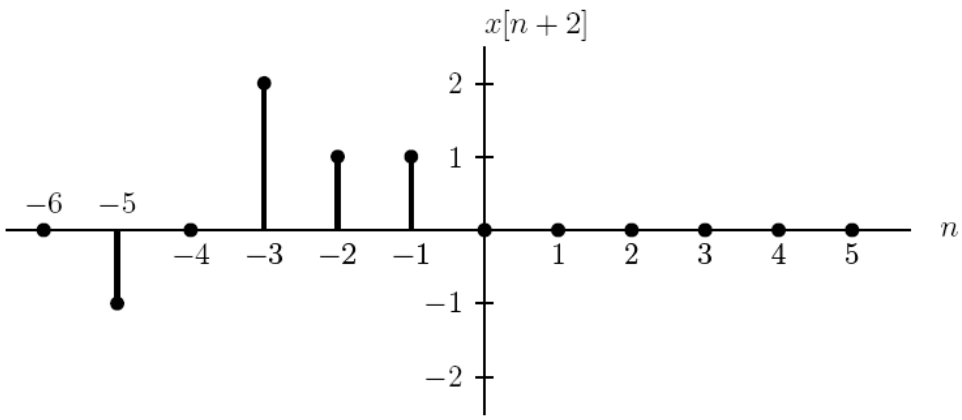
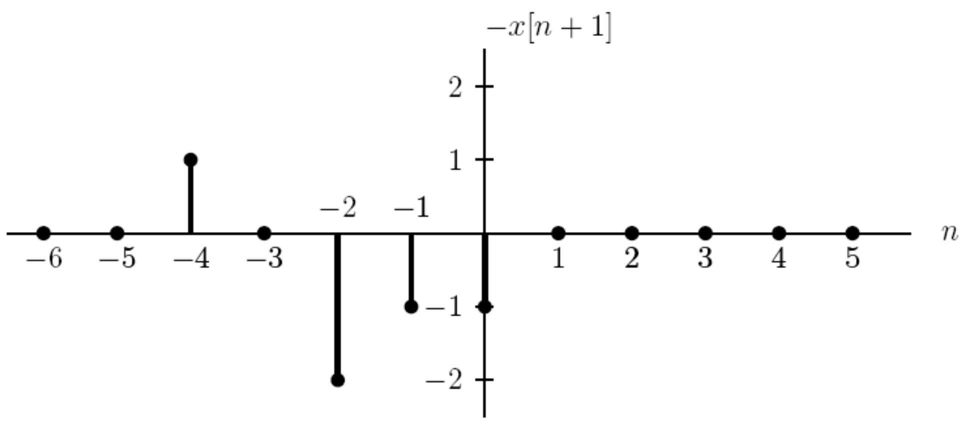
Problem 1

(a) With short duration DT sequences, it is often simplest to find their convolution by centering copies of one of the signals about each of the non-zero samples of the other signal and scaled by the value of the sample at that location. The result is the sum of all the shifted and scaled signals. Thus, $y[n]$ is given by the sum of the following signals.





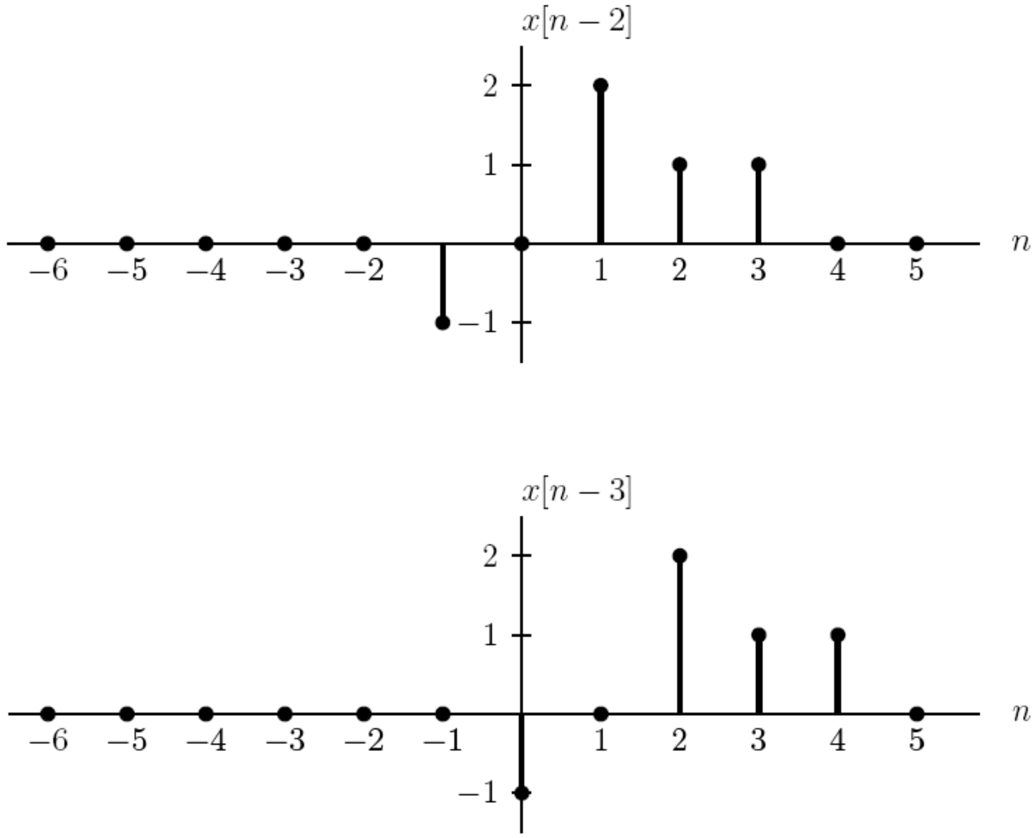


Figure 2.1.a.1: $x[n]$ **Scaled and shifted**

sum of these yields the following sequence for $y[n]$:

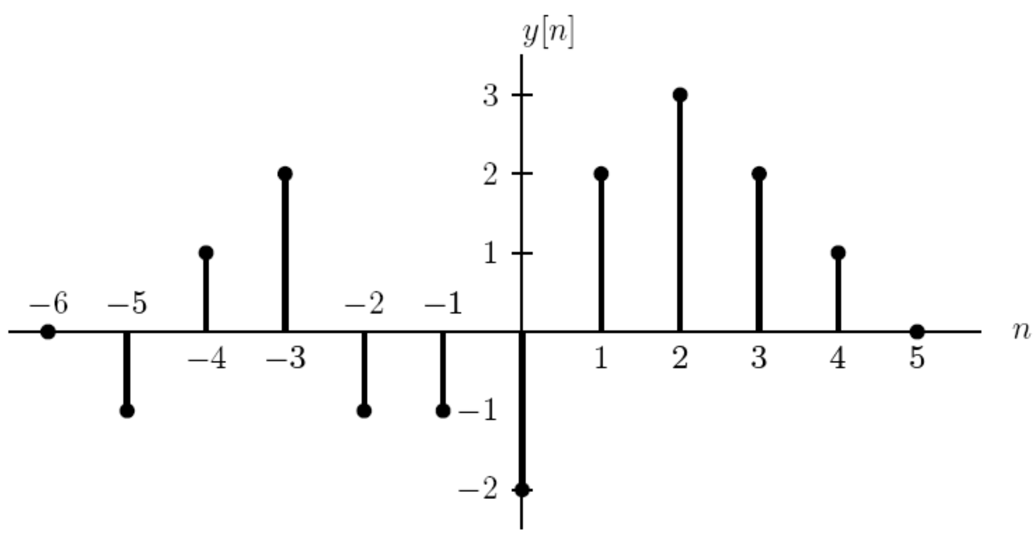


Figure 2.1.a.2: $y[n]$

(b) For this part, we can again use the shift and scale method since the sequence $x[n]$ is of a short duration as given below:

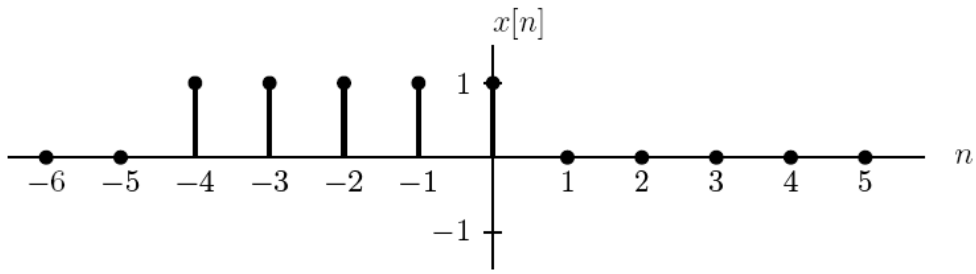


Figure 2.1.b: $x[n]$

Thus, we can write the output as a sum of scaled shifted inputs as follows:

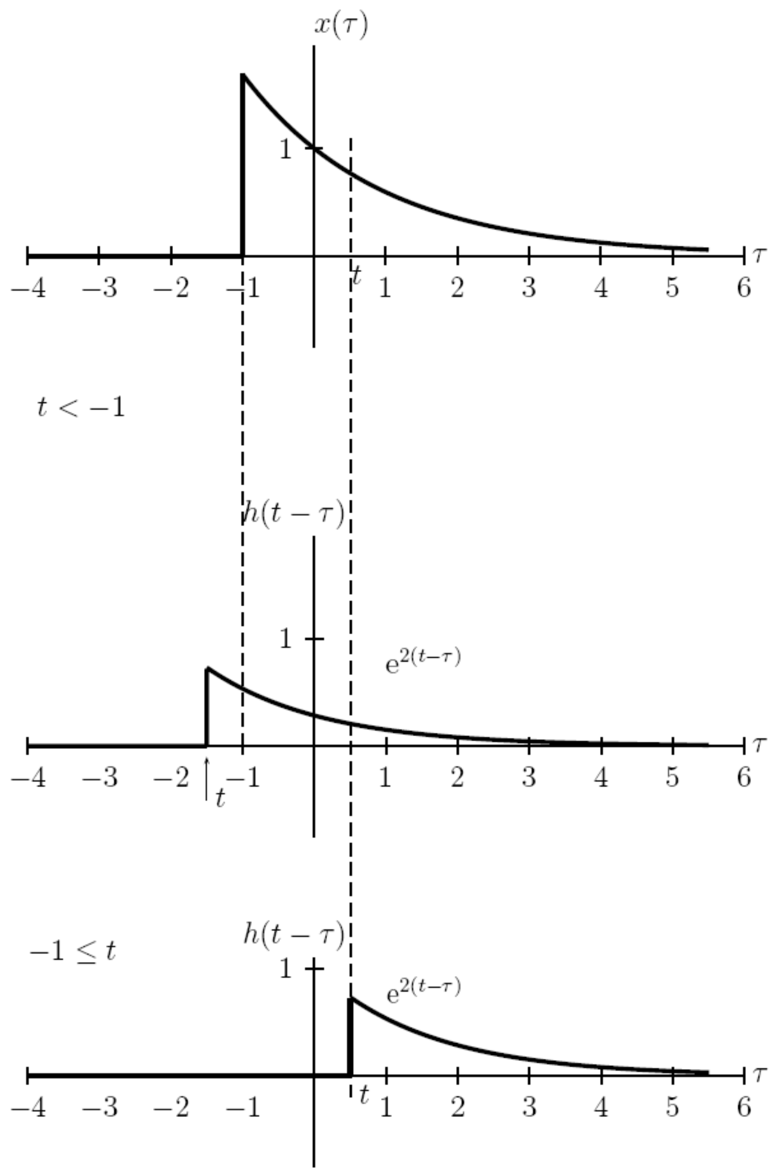
$$
\begin{aligned}
y[n] &= 2^n u[2-n] + 2^{n+1}u[1-n] + 2^{n+2}u[-n] + 2^{n+3}u[-n-1] + 2^{n+4}u[-n-2] \\
&= \sum_{k=0}^{4} 2^{n+k} u[2-n-k]
\end{aligned}
$$

Problem 2

(a) From the definition of the convolution, we have the following expression for the output $y(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Based on the given $x(t)$ and $h(t)$, we can break the integration up into 2 regions as illustrated in the diagram. The ranges are $t < -1$ and $t \geq -1$.



For the range $t < -1$, the region where $x(\tau)h(t-\tau)$ is non-zero is from $-1 \rightarrow \infty$. So, the expression for $y(t)$ is given by:

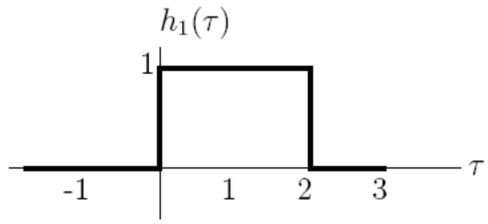
$$
\begin{aligned}
y(t) &= \int_{-1}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{-1}^{\infty} \mathrm{e}^{2(t-\tau)}\mathrm{e}^{-\tau}d\tau \\
&= \mathrm{e}^{2t}\int_{-1}^{\infty} \mathrm{e}^{-3\tau}d\tau = \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3\tau}\right]_{-1}^{\infty} \\
&= \frac{1}{3}\mathrm{e}^{2t+3}
\end{aligned}
$$

For the range $t \geq -1$, the $x(\tau)h(t-\tau)$ is non-zero for $\tau > t$. So the expression for $y(t)$ is given by:

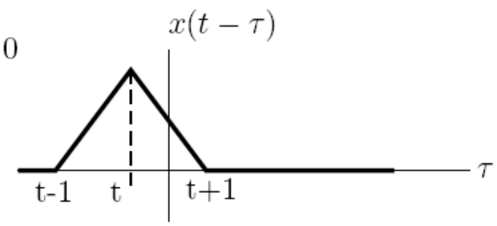
$$
\begin{aligned}
y(t) &= \int_{t}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{t}^{\infty} \mathrm{e}^{2(t-\tau)}\mathrm{e}^{-\tau}d\tau \\
&= \mathrm{e}^{2t}\int_{t}^{\infty} \mathrm{e}^{-3\tau}d\tau = \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3\tau}\right]_{t}^{\infty} \\
&= \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3t}\right] \\
&= \frac{1}{3}\mathrm{e}^{-t}
\end{aligned}
$$

(b) Here, we can break $h(t)$ up into $h(t) = h_1(t) + h_2(t)$ where $h_1(t)$ is the "box" part of $h(t)$ and $h_2(t)$ are the two impulses. Let $y_1(t)$ and $y_2(t)$ denote the result of convolving $x(t)$ with $h_1(t)$ and $h_2(t)$ respectively.

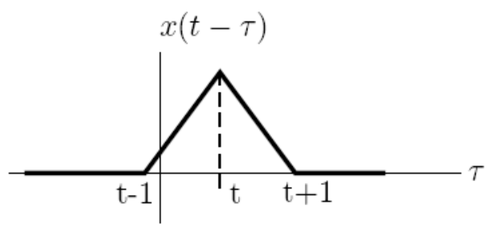
First let us compute $y_1(t)$. To do this, we fix $h_1(t)$ and flip and slide $x(t)$. The following figure illustrates the different regions of overlap.



$-1 < t < 0$



$0 \leq t < 1$



For the range $-1 < t < 0$, the result of the convolution is the area under the product of the two signals which is given by:

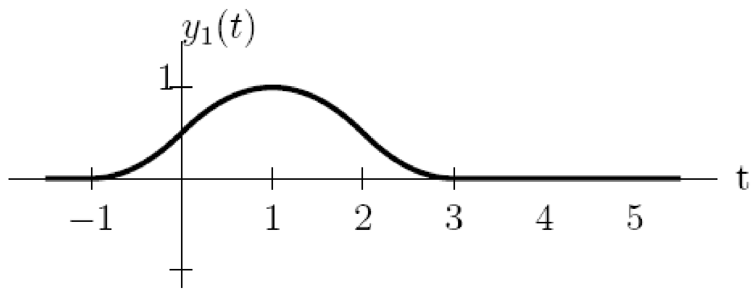
$$
\begin{aligned}
y_1(t) &= \frac{1}{2}(t+1)(t+1) \\
&= \frac{1}{2}(t^2 + 2t + 1)
\end{aligned}
$$

For the range $0 \leq t < 1$, the area under the product is given by:

$$
\begin{aligned}
y_1(t) &= t(1-t) + \frac{1}{2}t(1-(1-t)) + \frac{1}{2} \\
&= t - t^2 + \frac{1}{2}t^2 + \frac{1}{2} \\
&= \frac{1}{2}(1 + 2t - t^2)
\end{aligned}
$$

Now both $x(t)$ and $h_1(t)$ are symmetric signals which are symmetric about $t = 0$ and $t = 1$ respectively. Therefore, the convolution of the two is symmetric about $t = 1$.

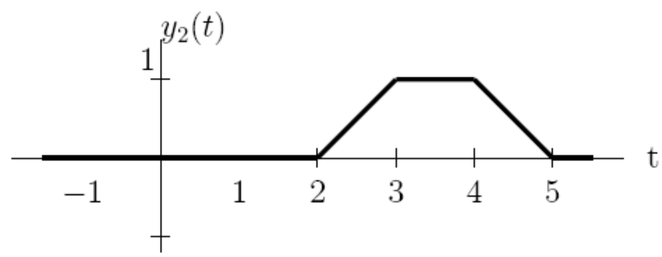
The plot for $y_1(t)$ looks like the following:



With the different regions of the curve as follows:

$$
\begin{aligned}
-1 < t < 0, \qquad & y_1(t) = \frac{1}{2}(t^2 + 2t + 1) \\
0 \le t < 1, \qquad & y_1(t) = \frac{1}{2}(1 + 2t - t^2) \\
1 \le t < 2, \qquad & y_1(t) = \frac{1}{2}(1 + 2t - t^2) \\
2 \le t < 3, \qquad & y_1(t) = \frac{1}{2}(t^2 - 6t + 9)
\end{aligned}
$$

The convolution with $h_2(t)$ is straightforward because it is a convolution with impulses. To do this, all we need is to center the triangle around both impulses and scale by the area under each impulse which in this case is 1. This gives the following plot for $y_2(t)$.



The final result is the sum of the two as follows:

The curved parts of the plot are given by the following expressions:

$$
\begin{aligned}
-1 < t < 0, \qquad & y(t) = \frac{1}{2}(t^2 + 2t + 1) \\
0 \leq t < 1, \qquad & y(t) = \frac{1}{2}(1 + 2t - t^2) \\
1 \leq t < 2, \qquad & y(t) = \frac{1}{2}(1 + 2t - t^2) \\
2 \leq t < 3, \qquad & y(t) = \frac{1}{2}(t^2 - 4t + 5)
\end{aligned}
$$

Problem 3

(a) Since the unit sample response is non-zero for $n < 0$, the system is not causal. For stability, we need to ensure that the impulse response is absolutely summable.

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |h[n]| &= \sum_{k=-\infty}^{3} 2^k \\ &= \sum_{k=\infty}^{-3} \left(\frac{1}{2}\right)^k \end{aligned}$$

which is finite. Thus, the system is stable

(b) Since $h(t)$ is 1 for $t < 0$, the system is not causal. For stability, the impulse response has to be absolutely integrable:

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} h(t) dt \qquad \text{since } h(t) \text{ is never negative} \\ &= \int_{-\infty}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \\ &= \int_{-\infty}^{0} u(1-t) dt + \int_{0}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \\ &= \int_{-\infty}^{0} 1 dt + \int_{0}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \end{aligned}$$

The first term on the r.h.s. of the equation integrates to ∞ but the second term is finite, which means the sum of the two terms is infinite. So, the system is not stable.

(c) This system is causal because the impulse response is zero for $n < 0$. For stability, the impulse response has to be absolutely summable.

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |h[k]| &= \sum_{k=0}^{\infty} h[k] + \sum_{k=-\infty}^{-1} -h[k] \\ &= \sum_{k=0}^{\infty} [1 - (0.99)^k] u[k] + \sum_{k=-\infty}^{-1} [1 - (0.99)^k] u[k] \\ &= \sum_{k=0}^{\infty} [1 - (0.99)^k] \\ &= \sum_{k=0}^{\infty} 1 - \sum_{k=0}^{\infty} (0.99)^k \end{aligned}$$

The second term on the r.h.s. is finite, as we know from power series, and the formulae we derived in problem set 1. The first term on the r.h.s. of the equation is infinite. So, the r.h.s. is infinite, which means the system is not stable.

(d) Since $h(t) = 0$ for all $t < 0$, this system is causal. Now, let's check for stability by taking the integral of the absolute value.

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} h(t)dt \qquad \text{since } h(t) \text{ is always positive} \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{15t}[u(t-1) - u(t-100)]dt \\
&= \int_{1}^{100} \mathrm{e}^{15t}dt \\
&= \frac{1}{15}\mathrm{e}^{15t}\Big|_{1}^{100} \\
&= \frac{1}{15}(\mathrm{e}^{1500} - \mathrm{e}^{15})
\end{aligned}$$

Which is finite. So, the system is stable.

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(c) The desired convolution is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{0}^{2} \sin(\pi\tau)h(t-\tau)d\tau. \end{aligned}$$

This gives us

$$y(t) = \begin{cases} 0, & t < 1 \\ (2/\pi)[1-\cos\{\pi(t-1)\}], & 1 < t < 3 \\ (2/\pi)[\cos\{\pi(t-3)\}-1], & 3 < t < 5 \\ 0, & 5 < t \end{cases}$$

(d) Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2),$$

where

$$h_1(t) = \begin{cases} 4/3, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Now,

$$y(t) = h(t) * x(t) = [h_1(t) * x(t)] - \frac{1}{3}x(t-2).$$

We have

$$h_1(t) * x(t) = \int_{t-1}^{t} \frac{4}{3}(a\tau + b)d\tau = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)].$$

Therefore,

$$y(t) = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)] - \frac{1}{3}[a(t-2)+b] = at + b = x(t).$$

(e) $x(t)$ periodic implies $y(t)$ periodic. $\therefore$ determine 1 period only. We have

$$y(t) = \begin{cases} \int_{t-1}^{-\frac{1}{2}} (t-\tau-1)d\tau + \int_{-\frac{1}{2}}^{t} (1-t+\tau)d\tau = \frac{1}{4} + t - t^2, & -\frac{1}{2} < t < \frac{1}{2} \\ \int_{t-1}^{\frac{1}{2}} (1-t+\tau)d\tau + \int_{\frac{1}{2}}^{t} (t-1-\tau)d\tau = t^2 - 3t + 7/4, & \frac{1}{2} < t < \frac{3}{2} \end{cases}$$

The period of $y(t)$ is 2.

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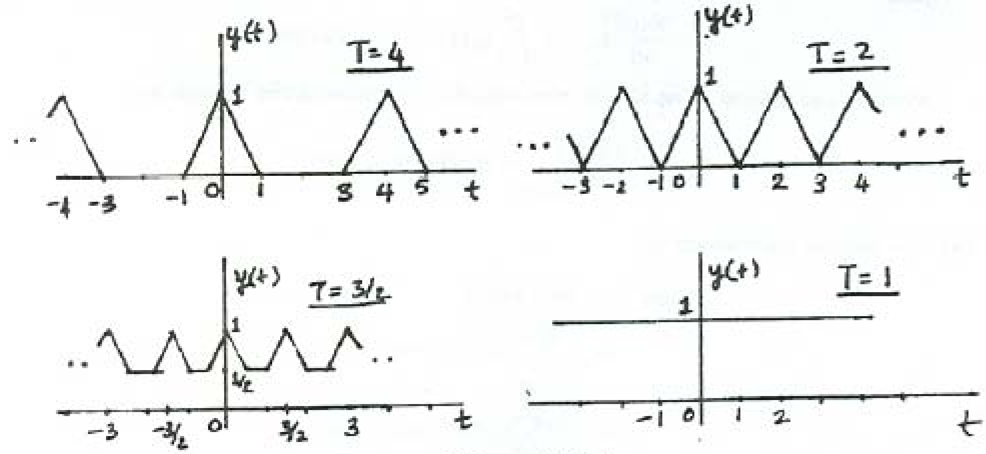

Figure S2.23

2.25

(a) We may write $x[n]$ as

$$x[n] = \left(\frac{1}{3}\right)^{|n|}.$$

Now, the desired convolution is

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{-1} (1/3)^{-k}(1/4)^{n-k}u[n-k+3] + \sum_{k=0}^{\infty}(1/3)^k(1/4)^{n-k}u[n-k+3] \\ &= (1/12)\sum_{k=0}^{\infty}(1/3)^k(1/4)^{n+k}u[n+k+4] + \sum_{k=0}^{\infty}(1/3)^k(1/4)^{n-k}u[n-k+3] \end{aligned}$$

By consider each summation in the above equation separately, we may show that

$$y[n] = \begin{cases} (12^4/11)3^n, & n < -4 \\ (1/11)4^4, & n = -4 \\ (1/4)^n(1/11) + -3(1/4)^n + 3(256)(1/3)^n, & n \geq -3 \end{cases}.$$

(b) Now consider the convolution

$$y_1[n] = [(1/3)^n u[n]] * [(1/4)^n u[n+3]].$$

We may show that

$$y_1[n] = \begin{cases} 0, & n < -3 \\ -3(1/4)^n + 3(256)(1/3)^n, & n \geq -3 \end{cases}.$$

Also, consider the convolution

$$y_2[n] = [(3)^n u[-n-1]] * [(1/4)^n u[n+3]].$$

We may show that

$$y_2[n] = \begin{cases} (12^4/11)3^n, & n < -4 \\ (1/4)^n(1/11), & n \geq -3 \end{cases}.$$

Clearly, $y_1[n] + y_2[n] = y[n]$ obtained in the previous part.

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