In The Name of God, The Merciful, The Compassionate

Signals and Systems

Department of Computer Engineering

Sharif University of Technology

Fall 2007 – CE 242

Midterm Exam

Name:

Student ID:

Instructions:

Follow all instructions carefully!
Do not forget to write your name and student ID.
This is a 100 minute exam containing five problems totaling 100 points.
You may not use any notes, books or other references.
You may use the Fact Sheet only as a reference.
You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFS

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T} \end{aligned}$$

- CTFT

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(\omega/(2\pi))$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega+a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$\begin{aligned} X_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ x(n) &= \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- DTFT

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT) \\ S(\omega) &= Y(\omega T) \end{aligned}$$

Problem1

A)

For each of the three systems below specify whether or not the system is (1) linear, (2) time-invariant, (3) causal, (4) stable. The system input is $x[n]$ and the system output is $y[n]$.

a) $y[n] = x[2n]$

b) $y[n] = x[n] + x[n-1]$

c) $y[n] = (x[-|n|])^2$

B)

a) Determine the difference equation relating the input and output of the LTI system described by the following frequency response:

$$H(e^{j\omega}) = \frac{1 + \frac{1}{2}e^{-j\omega}}{1 - 2e^{-j\omega}}$$

b) Is the system in (a) stable?

c) Is the system in (a) causal?

Problem 2

Consider the discrete-time signal $f : \mathbb{Z} \rightarrow \mathbb{R}$, characterized as follows:

$$\forall m \in \mathbb{Z}, \quad f(m) = \begin{cases} 1 & m = 0, 1, 2 \\ 0 & elsewhere. \end{cases}$$

You can tackle the two parts of this problem independently.

a) A related signal $p : \mathbb{Z} \rightarrow \mathbb{R}$, results as follows:

$$\forall m \in \mathbb{Z}, \quad p(m) = \frac{1}{2}[1 + (-1)^m] f(m).$$

Provide a well-labeled sketch of the signal p.

b) A related signal $q : \mathbb{Z} \rightarrow \mathbb{R}$, results from the convolution of the signal f with itself; this is written as $q = f * f, or\ q(n) = (f * f)(n), \forall n \in \mathbb{Z}.$. In particular, the signal q satisfies the following *convolution sum*:

$$\forall n \in \mathbb{Z}, \quad q(n) = \sum_{m=-\infty}^{\infty} f(m) f(n-m).$$

Provide a well-labeled sketch of graph(q).

Problem 3

Consider an LTI system with frequency response as

$$H(e^{j\omega}) = \frac{1 - \frac{7}{4}e^{-j\omega} - \frac{1}{2}e^{-2j\omega}}{1 + \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega}}$$

a) Find 2 different block diagrams associated with this system, and hence show that it is not unique. (Hint : Try to factorize H and write it as combination of forms which you know their block diagram).
b) Find the impulse response of the system.
c) Verify whether the system is stable.
d) Describe the functionality of the filter as either low, high or band pass and justify your answer.

Problem 4

Consider the system described by the equation below:

$$y[n]=\frac{1}{2}(x[n]-x[n-1])-y[n-1]$$

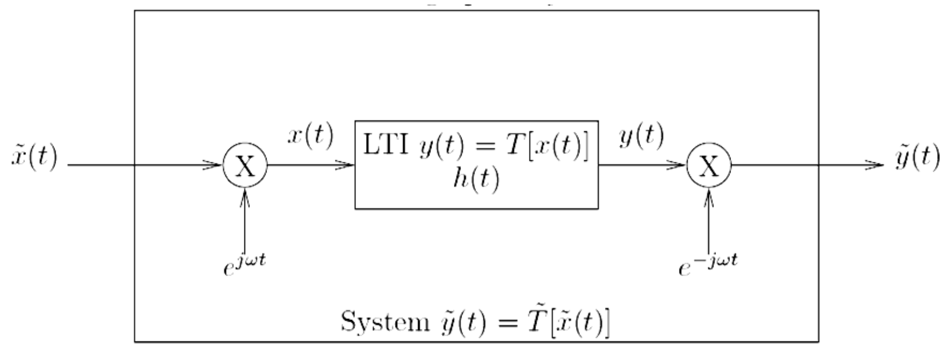
a. Find a simple expression for the frequency response H($e^{j\omega}$) for this system.
b. Find an expression for the magnitude |H($e^{j\omega}$)| of the frequency response, and sketch it. Be sure to label your axes.
c. Find an expression for the phase H($e^{j\omega}$) of the frequency response, and sketch it. Be sure to label your axes.

Problem 5

LTI Systems

Consider a continuous time LTI system y(t) = T[x(t)] with input x(t), output y(t), and impulse response h(t). Furthermore, consider the system $\tilde{y}(t) = \tilde{T}[\tilde{x}(t)]$ where $x(t) = e^{jwt}\tilde{x}(t)$ and $\tilde{y}(t) = e^{-jwt}y(t)$ and w is a constant.

The figure below illustrates the situation graphically.



a. Write an expression for y(t) in terms of the functions x(t) and h(t).
b. Write an expression for $\tilde{y}(t)$ in terms of the functions $\tilde{x}(t)$ and h(t).
c. Either prove that the system $\tilde{y}(t) = \tilde{T}[\tilde{x}(t)]$ is LTI, or prove it is not LTI.
d. Find the impulse response of the system $\tilde{T}[\tilde{x}(t)]$