Digital Media Laboratory
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Signals & Systems

Time and Frequency Characterization of

Signals and Systems

Adapted from: Lecture notes from MIT and Concordia University

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Magnitude and Phase of FT, and Parseval Relation

CT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$$

Parseval Relation $\displaystyle\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} |X(j\omega)|^2\, d\omega$

Magnitude and Phase of FT, and Parseval Relation

DT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X(e^{j\omega}) = |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega})}$$

Parseval Relation $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{2\pi} \frac{1}{2\pi} |X(e^{j\omega})|^2\, d\omega$$

Effects of Phase

- *Not* on signal energy distribution as a function of frequency
- Can have dramatic effect on signal shape/character
  - Constructive/Destructive interference
- Is that important?
  - Depends on the signal and the context

Effects of Phase

Demo:

Effect of phase on Fourier Series

Applet 17

Log-Magnitude and Phase

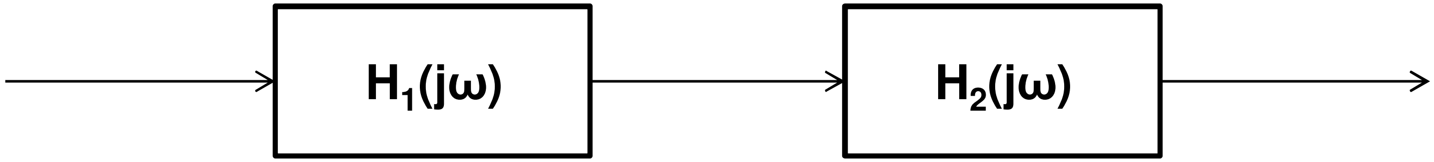
$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

$$|Y(j\omega)| = |H(j\omega)| \, . \, |X(j\omega)|$$

$$\log |Y(j\omega)| = \log |H(j\omega)| + \log |X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Log-Magnitude and Phase



$$\log |H(j\omega)| = \log |H_1(j\omega)| + \log |H_2(j\omega)|$$

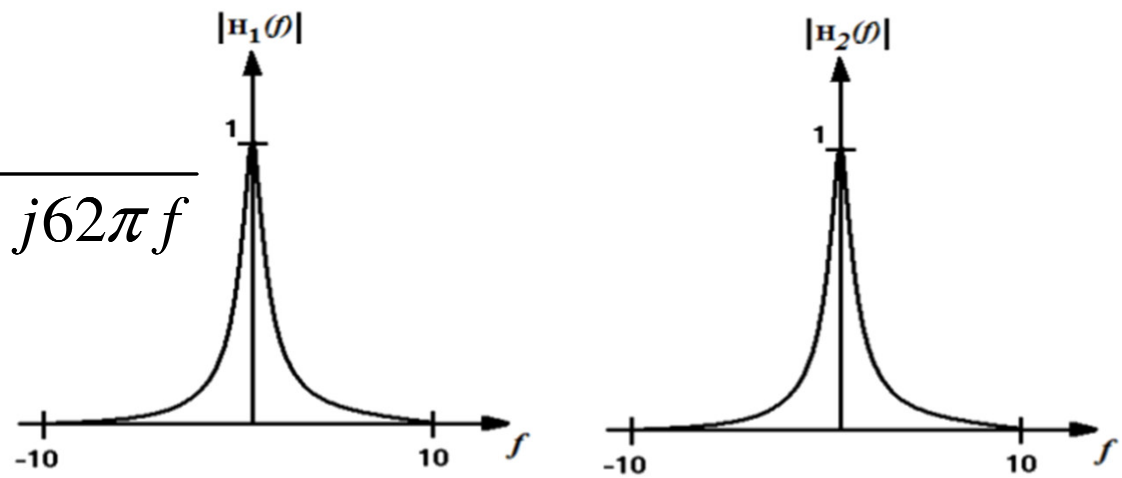
$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

Bode Diagram

- Although linear plots, e.g., H(jw) versus w, of frequency responses are accurate, however, they do not always reveal important system behavior
- The plots of the two quite different-looking frequency responses look identical

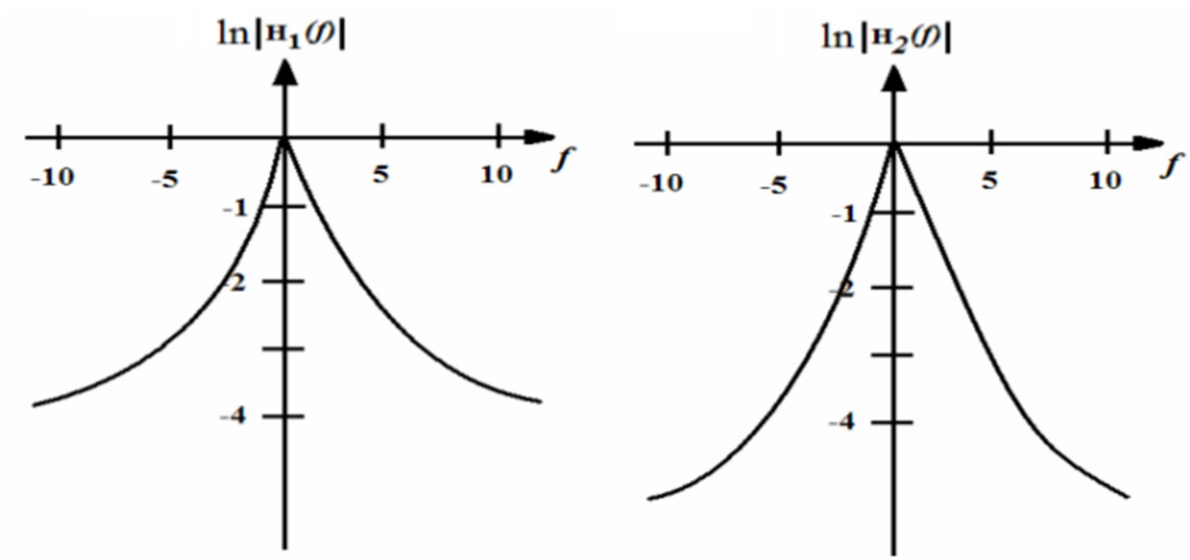
$$H_1(f) = \frac{1}{j2\pi f + 1}$$

$$H_2(f) = \frac{30}{30 - 4\pi^2 f^2 + j62\pi f}$$



Bode Diagram

- If we plot the logarithm of magnitude instead of the magnitude, the difference can be seen more easily



Log-magnitude plots of the two frequency responses

Bode diagram

- A more common way of displaying frequency response is the Bode diagram or Bode plot
  - A log-magnitude plot is logarithmic in one dimension
  - A Bode diagram is logarithmic in both dimensions
- A bode diagram is a plot of the logarithm of the magnitude of a frequency response against a logarithmic frequency scale
  - log |H(jw)| versus scaled w
- In a bode diagram, |H(jw)| is converted to a logarithmic scale using the unit decibel (dB)

Plotting Log-Magnitude and Phase

- For real-valued signals and systems

$$\left.\begin{aligned} |H(-j\omega)| &= |H(j\omega)| \\ \angle H(-j\omega) &= -\angle H(j\omega) \end{aligned}\right\} \Rightarrow$$

Plot for $\omega \geq 0$, often with a *logarithmic* scale for frequency in CT

- In DT, need only plot for $0 \leq \omega \leq \pi$ (with *linear* scale)
- For historical reasons, log-magnitude is usually plotted in units of *decibels* (dB):

$$1\,\text{bel} = 10\,\text{decibels} = \frac{\text{output power}}{\text{input power}} = 10$$

Plotting Log-Magnitude and Phase

power magnitude

$$10\log_{10}|H(j\omega)|^2 = 20log_{10}|H(j\omega)|$$

$$\left|H(j\omega)\right| = 1 \quad \rightarrow \quad 0dB$$

$$\left|H(j\omega)\right| = \sqrt{2} \quad \rightarrow \quad \sim 3dB$$

$$\left|H(j\omega)\right| = 2 \quad \rightarrow \quad \sim 6dB$$

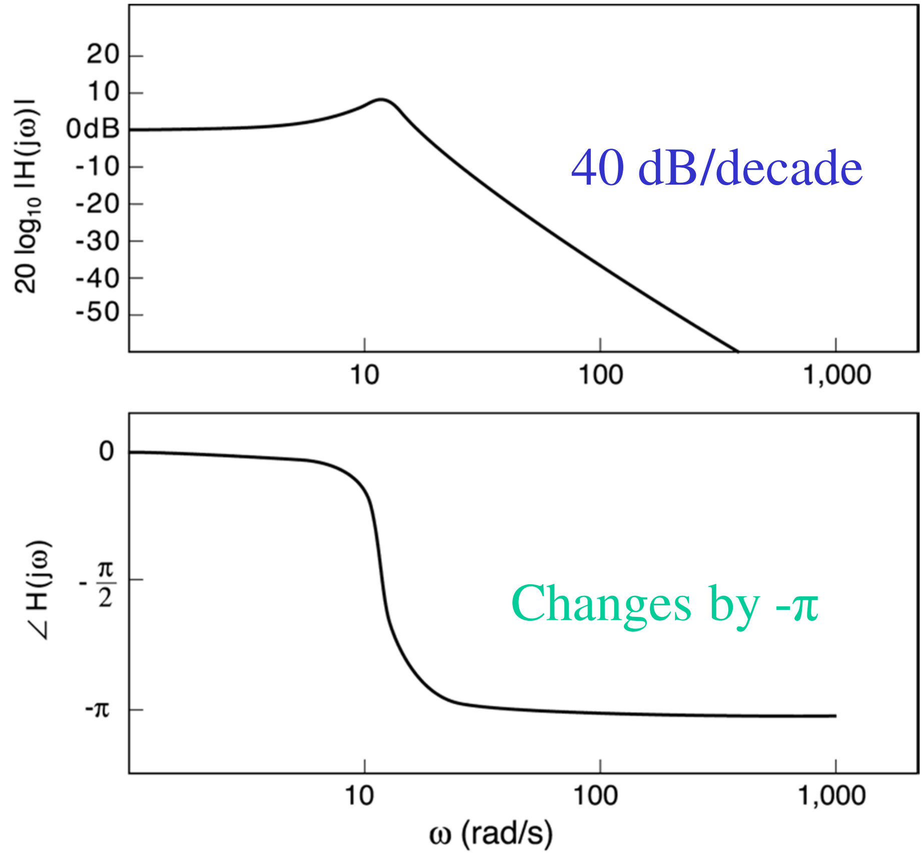
$$\left|H(j\omega)\right| = 10 \quad \rightarrow \quad 20dB$$

$$\left|H(j\omega)\right| = 100 \quad \rightarrow \quad 40dB$$

So… 20 dB or 2 *bels*
= 10 amplitude gain
= 100 power gain

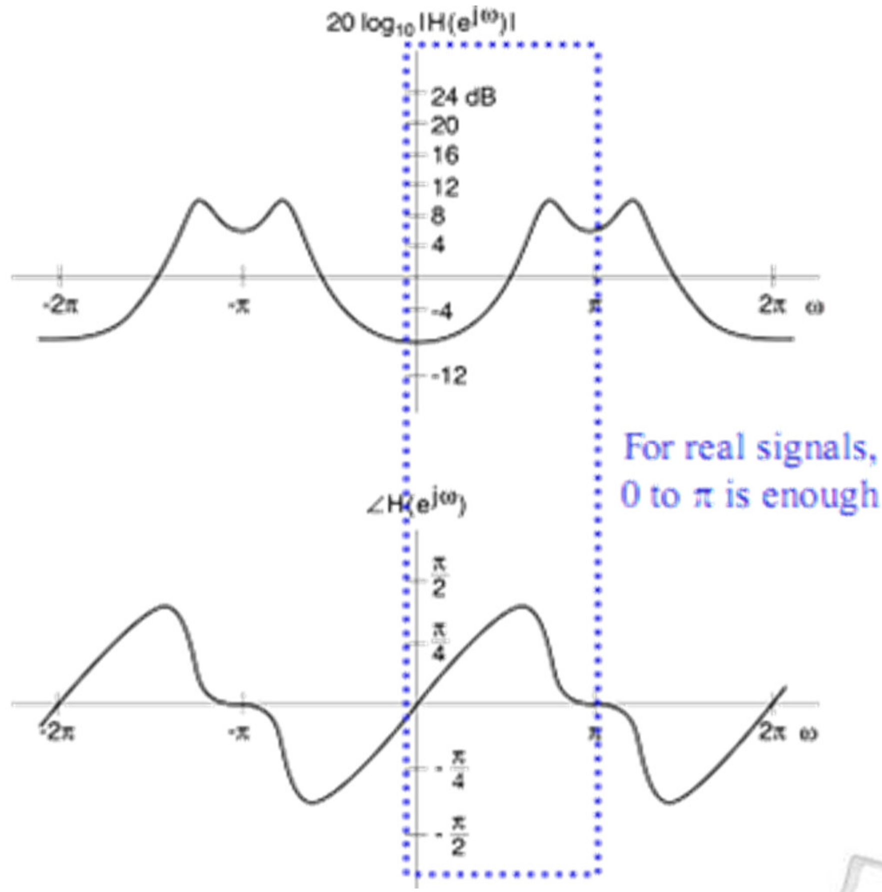
A Typical Bode plot for a second-order CT system

$$20\log|H(j\omega)| \text{ and } \angle H(j\omega) \text{ vs. } \log\omega$$



A Typical Bode plot of the magnitude and phase for a second-order DT frequency response

$$20\log|H(e^{j\omega})| \text{ and } \angle H(e^{j\omega}) \text{ vs. } \omega$$



Linear Phase

CT

$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

$$H(j\omega) = e^{-j\omega\alpha} \quad \Rightarrow \quad |H(j\omega)| = 1,\ \angle H(j\omega) = -\alpha\omega \text{ (Linear in } \omega)$$

$$\mathrm{Y}(\mathrm{j}\omega) = e^{-j\omega\alpha} X(j\omega) \quad \overset{\text{time-shift}}{\longleftrightarrow} \quad y(t) = x(t-\alpha)$$

Result: Linear phase ⇔ simply a rigid shift in time, *no* distortion
Nonlinear phase ⇔ distortion as well as shift

Linear Phase

DT

$$x[n] \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y[n]$$

$$y[n] = x[n-n_0] \xleftrightarrow{\mathfrak{F}} Y\left(e^{j\omega}\right) = e^{-j\omega n_0} X\left(e^{j\omega}\right)$$

$$H\left(e^{j\omega n_0}\right) = e^{-j\omega n_0} \Rightarrow \left|H\left(e^{j\omega}\right)\right| = 1,\ \angle H\left(e^{j\omega}\right) = -n_0\omega$$

Question:

What about $H(e^{j\omega}) = e^{-j\omega\alpha}$, $\alpha \neq$ integer?

All-Pass Systems

CT

$$|H(j\omega)| = 1$$

$$H(j\omega) = e^{-j\alpha\omega} \qquad \text{Linear phase}$$

$$H(j\omega) = \frac{\alpha - j\omega}{\alpha + j\omega} \qquad \text{Nonlinear phase}$$

$$|H(j\omega)| = \sqrt{\frac{\alpha^2 + \omega^2}{\alpha^2 + \omega^2}}$$

All-Pass Systems

DT

$$|H(e^{j\omega})| = 1$$

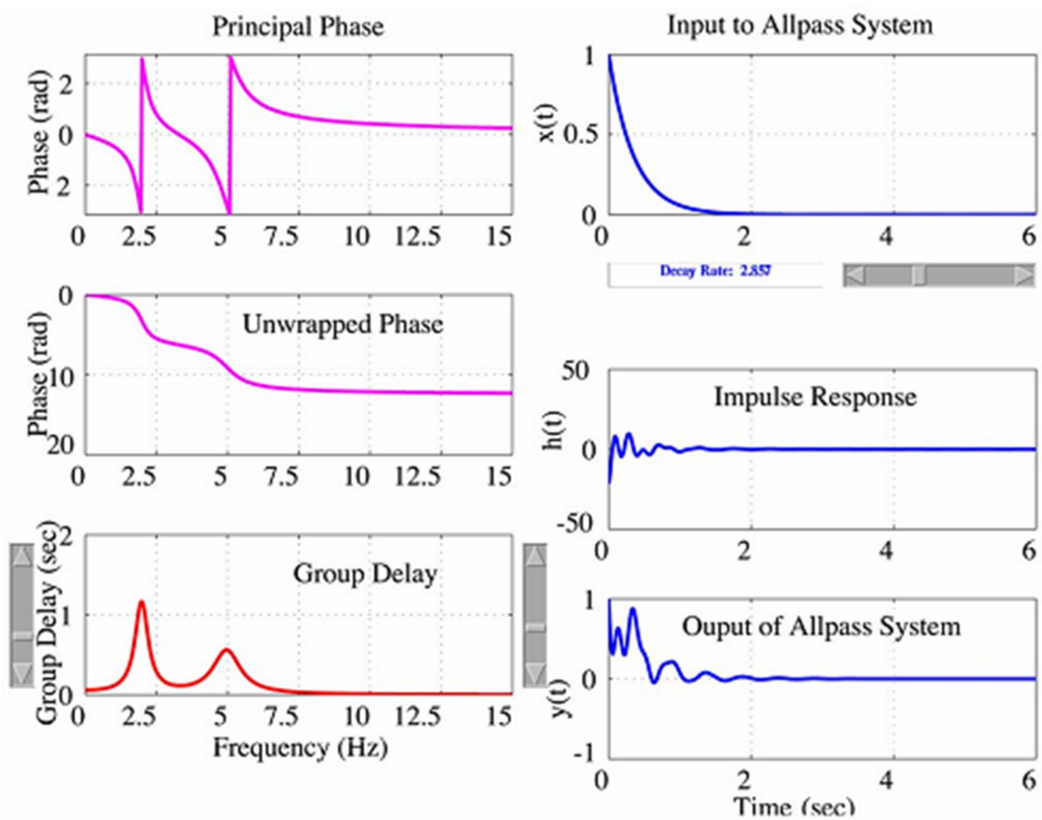
$$H(j\omega) = e^{-j\alpha\omega n_0}$$ Linear phase

$$H(j\omega) = \frac{1 - 1/2 . e^{j\omega}}{1 - 1/2 . e^{-j\omega}}$$ Nonlinear phase

$$|H(e^{j\omega})| = \sqrt{\frac{(1 - 1/2 . cos\omega)^2 + (1/2 . sin\omega)^2}{(1 - 1/2 . cos\omega)^2 + (1/2 . sin\omega)^2}} = 1$$

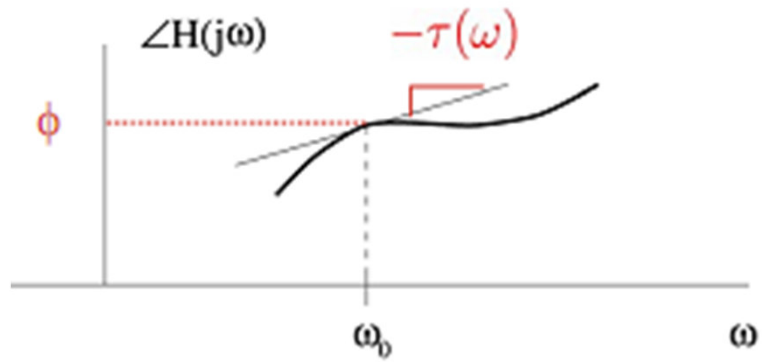
Sample

Impluse response and output of an all-pass system with nonlinear phase



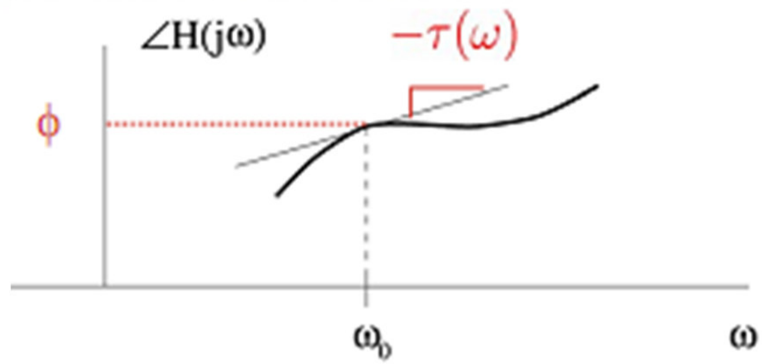
Group Delay

How do we think about signal delay when the phase is nonlinear?



When the signal is narrow-band and concentrated near ω_0, $\angle H(j\omega)$ ~ linear with ω near ω_0, then $-\frac{d\angle H(j\omega)}{d\omega}$ instead of $\frac{\angle H(j\omega)}{d\omega}$ reflects the time delay.

Group Delay



For frequencies "near" ω_0

$$\angle H(j\omega) \approx \angle H(j\omega_0) - \tau(\omega_0)(\omega - \omega_0) = \phi - \tau(\omega_0).\omega$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\} = \text{Group Delay}$$

$$\Downarrow$$

For ω near ω_0

$$H(j\omega) \approx |H(j\omega_0)| e^{j\phi} e^{-j\tau(\omega_0)\omega}$$

$$\Rightarrow \qquad e^{j\omega t} \rightarrow\sim |H(j\omega)| e^{j\phi} e^{j\omega(t-\tau(\omega_0))}$$

Linear phase / Group delay

- **Group delay** is a measure of the transit time of a signal through a system (or device) versus frequency
- **Linear phase** is a property of a filter
  - The phase response of the filter is a linear function of frequency
- Linear phase filter has the property of the true time delay
- A linear phase filter has constant group delay
  - All frequency components of a signal have equal delay times
  - There is no distortion due of select frequencies
- A filter with **non-linear** phase has a group delay that varies with frequency, resulting in phase distortion