Digital Media Laboratory
Advanced Information & Communication Technology Center
Sharif University of Technology

# Signals & Systems

## Time and Frequency Characterization of Signals and Systems

Adapted from: Lecture notes from MIT and Concordia University

Dr. Hamid R. Rabiee

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# Magnitude and Phase of FT, and Parseval Relation

## CT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$$

Parseval Relation $\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} |X(j\omega)|^2\, d\omega$

# Magnitude and Phase of FT, and Parseval Relation

## DT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X(e^{j\omega}) = |X(e^{j\omega})|\,e^{j\angle X(e^{j\omega})}$$

Parseval Relation $$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \int_{2\pi}\frac{1}{2\pi}|X(e^{j\omega})|^2\,d\omega$$

# Effects of Phase

- *Not* on signal energy distribution as a function of frequency
- Can have dramatic effect on signal shape/character
  - Constructive/Destructive interference
- Is that important?
  - Depends on the signal and the context

# Effects of Phase

## Demo:

Effect of phase on Fourier Series

**Applet 17**

# Log-Magnitude and Phase

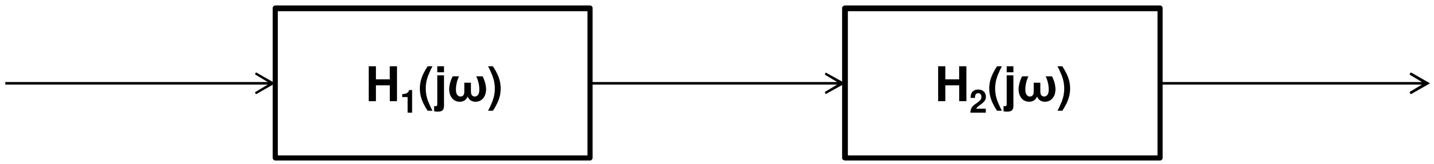
$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

$$|Y(j\omega)| = |H(j\omega)| . |X(j\omega)|$$

$$\log |Y(j\omega)| = \log |H(j\omega)| + \log |X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

# Log-Magnitude and Phase




$$\log|H(j\omega)| = \log|H_1(j\omega)| + \log|H_2(j\omega)|$$

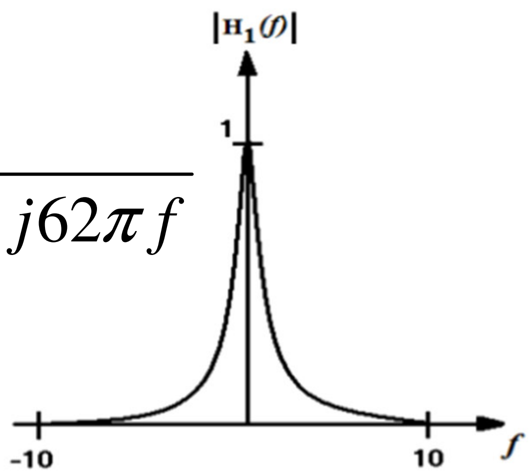
$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

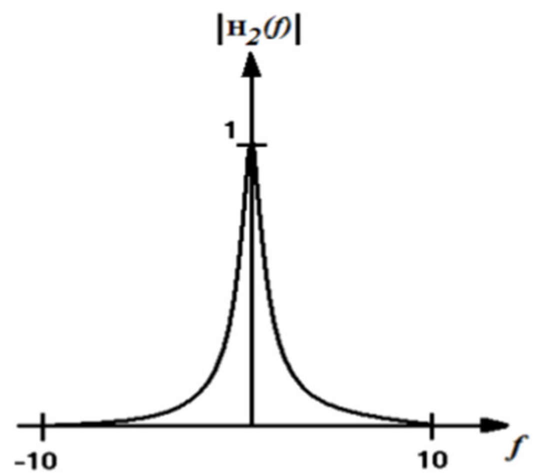
# Bode Diagram

- Although linear plots, e.g., H(jw) versus w, of frequency responses are accurate, however, they do not always reveal important system behavior
- The plots of the two quite different-looking frequency responses look identical

$$H_1(f) = \frac{1}{j2\pi f + 1}$$

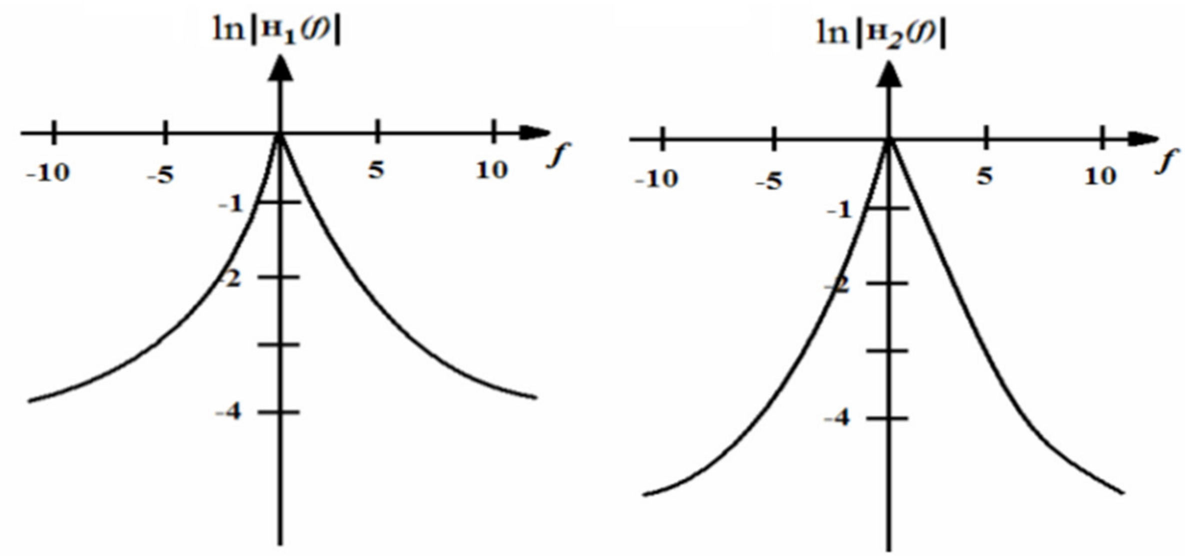
$$H_2(f) = \frac{30}{30 - 4\pi^2 f^2 + j62\pi f}$$



# Bode Diagram

- If we plot the logarithm of magnitude instead of the magnitude, the difference can be seen more easily




Log-magnitude plots of the two frequency responses

# Bode diagram

- A more common way of displaying frequency response is the Bode diagram or Bode plot
  - A log-magnitude plot is logarithmic in one dimension
  - A Bode diagram is logarithmic in both dimensions
- A bode diagram is a plot of the logarithm of the magnitude of a frequency response against a logarithmic frequency scale
  - log |H(jw)| versus scaled w
- In a bode diagram, |H(jw)| is converted to a logarithmic scale using the unit decibel (dB)

# Plotting Log-Magnitude and Phase

- **For real-valued signals and systems**

$$\left.\begin{aligned} |H(-j\omega)| &= |H(j\omega)| \\ \angle H(-j\omega) &= -\angle H(j\omega) \end{aligned}\right\} \Rightarrow$$

**Plot for $\omega \geq 0$, often with a *logarithmic* scale for frequency in CT**

- **In DT, need only plot for $0 \leq \omega \leq \pi$ (with *linear* scale)**

- **For historical reasons, log-magnitude is usually plotted in units of *decibels* (dB):**

$$1\,\text{bel} = 10\,\text{decibels} = \frac{\text{output power}}{\text{input power}} = 10$$

# Plotting Log-Magnitude and Phase

power magnitude

$$10\log_{10}|H(j\omega)|^2 = 20log_{10}|H(j\omega)|$$

$$\left|H(j\omega)\right| = 1 \quad \rightarrow \quad 0dB$$

$$\left|H(j\omega)\right| = \sqrt{2} \quad \rightarrow \quad \sim 3dB$$

$$\left|H(j\omega)\right| = 2 \quad \rightarrow \quad \sim 6dB$$

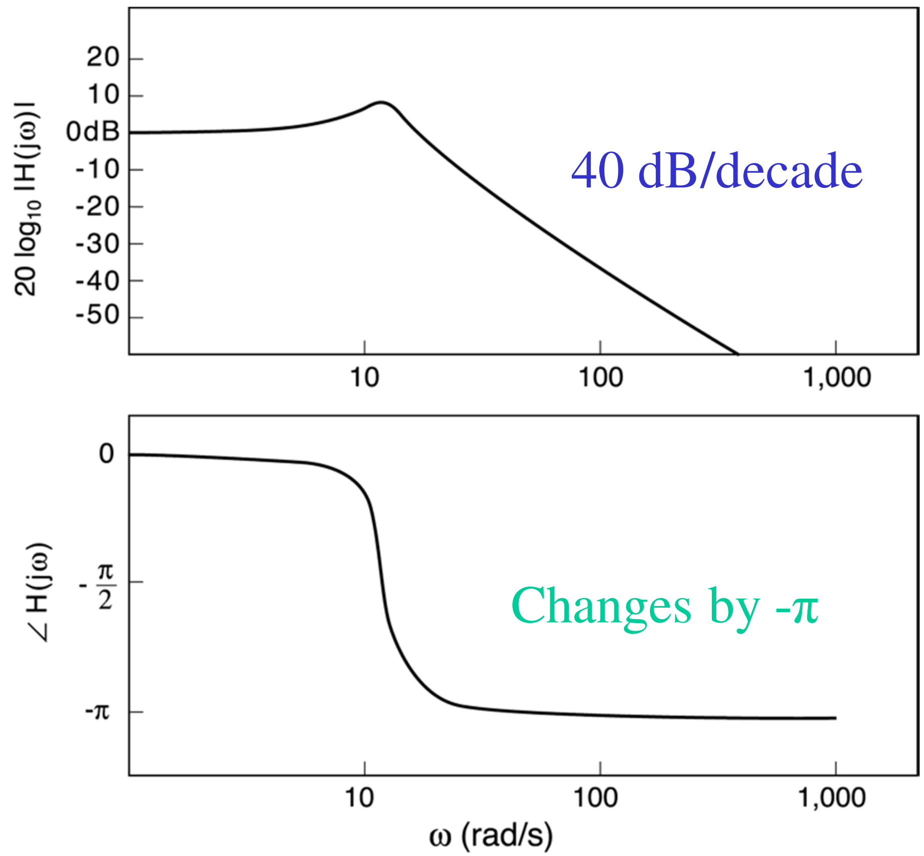
$$\left|H(j\omega)\right| = 10 \quad \rightarrow \quad 20dB$$

$$\left|H(j\omega)\right| = 100 \quad \rightarrow \quad 40dB$$

**So… 20 dB or 2 *bels***
**= 10 amplitude gain**
**= 100 power gain**

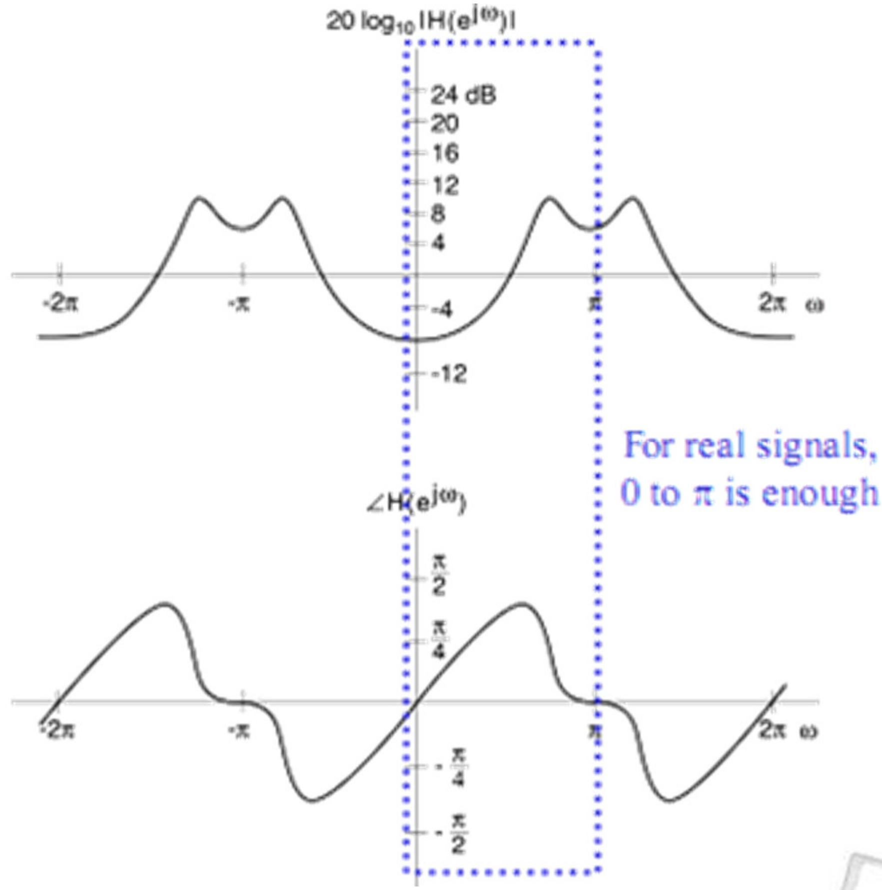
# A Typical Bode plot for a second-order CT system

$$20\log|H(j\omega)| \text{ and } \angle H(j\omega) \text{ vs. } \log\omega$$



# A Typical Bode plot of the magnitude and phase for a second-order DT frequency response

$$20\log|H(e^{j\omega})| \text{ and } \angle H(e^{j\omega}) \text{ vs. } \omega$$



# Linear Phase

## CT

$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

$$H(j\omega) = e^{-j\omega\alpha} \quad \Rightarrow \quad |H(j\omega)| = 1,\ \angle H(j\omega) = -\alpha\omega \text{ (Linear in } \omega\text{)}$$

$$\mathrm{Y}(\mathrm{j}\omega) = e^{-j\omega\alpha} X(j\omega) \quad \xleftrightarrow{\text{time-shift}} \quad y(t) = x(t-\alpha)$$

**Result: Linear phase ⇔ simply a rigid shift in time, *no* distortion**
**Nonlinear phase ⇔ distortion as well as shift**

# Linear Phase

## DT

$$x[n] \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y[n]$$

$$y[n] = x[n-n_0] \xleftrightarrow{\mathfrak{F}} Y\left(e^{j\omega}\right) = e^{-j\omega n_0} X\left(e^{j\omega}\right)$$

$$H\left(e^{j\omega n_0}\right) = e^{-j\omega n_0} \Rightarrow \left|H\left(e^{j\omega}\right)\right| = 1,\ \angle H\left(e^{j\omega}\right) = -n_0\omega$$

**Question:**

**What about $H(e^{j\omega}) = e^{-j\omega\alpha}$, $\alpha \neq$ integer?**

# All-Pass Systems

## CT

$$|H(j\omega)| = 1$$

$$H(j\omega) = e^{-j\alpha\omega}$$ Linear phase

$$H(j\omega) = \frac{\alpha - j\omega}{\alpha + j\omega}$$ Nonlinear phase

$$|H(j\omega)| = \sqrt{\frac{\alpha^2 + \omega^2}{\alpha^2 + \omega^2}}$$

# All-Pass Systems

## DT

$$|H(e^{j\omega})| = 1$$

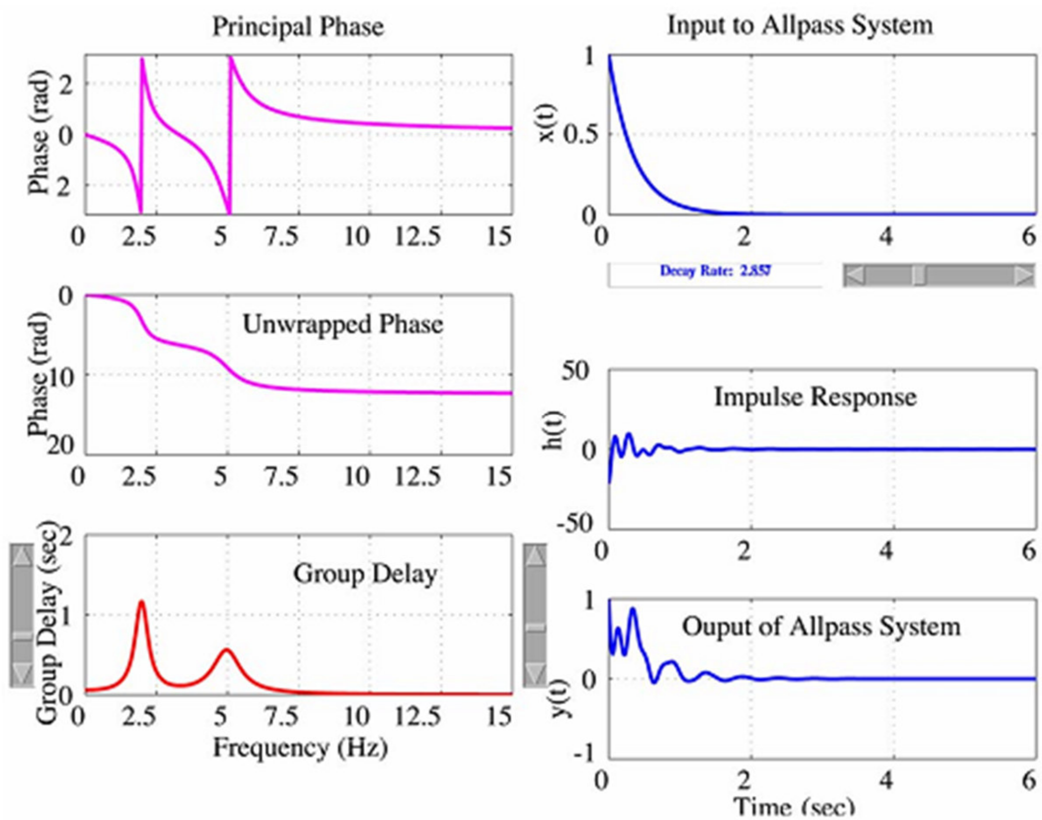
$$H(j\omega) = e^{-j\alpha\omega n_0}$$ Linear phase

$$H(j\omega) = \frac{1 - 1/2.e^{j\omega}}{1 - 1/2.e^{-j\omega}}$$ Nonlinear phase

$$|H(e^{j\omega})| = \sqrt{\frac{(1 - 1/2.cos\omega)^2 + (1/2.sin\omega)^2}{(1 - 1/2.cos\omega)^2 + (1/2.sin\omega)^2}} = 1$$

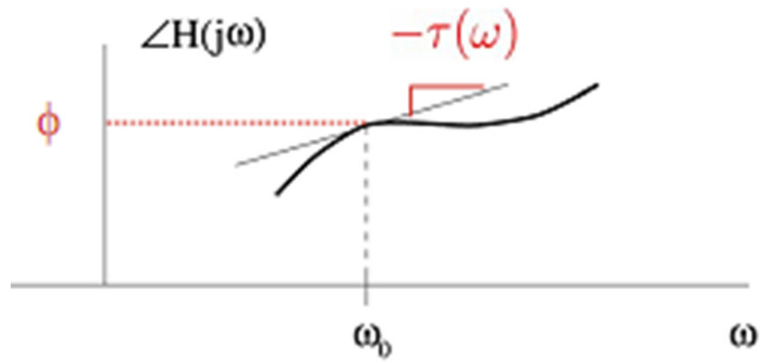
# Sample

**Impluse response and output of an all-pass system with nonlinear phase**



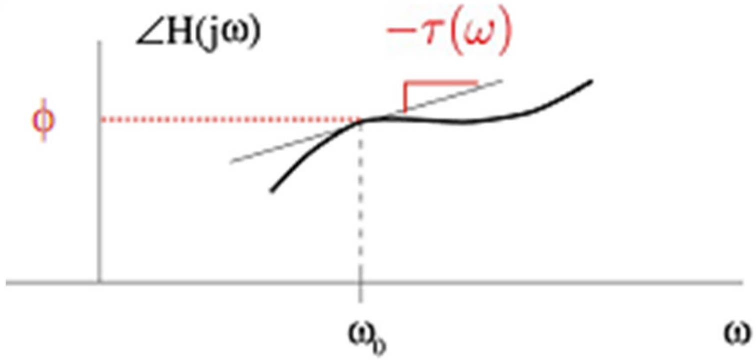
# Group Delay

**How do we think about signal delay when the phase is nonlinear?**




**When the signal is narrow-band and concentrated near $\omega_0$, $\angle H(j\omega) \sim$ linear with $\omega$ near $\omega_0$, then $-\frac{d\angle H(j\omega)}{d\omega}$ instead of $\frac{\angle H(j\omega)}{d\omega}$ reflects the time delay.**

# Group Delay




For frequencies "near" $\omega_0$

$$\angle H(j\omega) \approx \angle H(j\omega_0) - \tau(\omega_0)(\omega - \omega_0) = \phi - \tau(\omega_0).\omega$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\} = \text{Group Delay}$$

$$\Downarrow$$

For $\omega$ near $\omega_0$

$$H(j\omega) \approx |H(j\omega_0)| e^{j\phi} e^{-j\tau(\omega_0)\omega}$$

$$\Rightarrow \qquad e^{j\omega t} \rightarrow \sim |H(j\omega)| e^{j\phi} e^{j\omega(t - \tau(\omega_0))}$$

# Linear phase / Group delay

- **Group delay** is a measure of the transit time of a signal through a system (or device) versus frequency
- **Linear phase** is a property of a filter
  - The phase response of the filter is a linear function of frequency
- Linear phase filter has the property of the true time delay
- A linear phase filter has constant group delay
  - All frequency components of a signal have equal delay times
  - There is no distortion due of select frequencies
- A filter with **non-linear** phase has a group delay that varies with frequency, resulting in phase distortion