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Signals & Systems

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Review on Bases of Vector Spaces

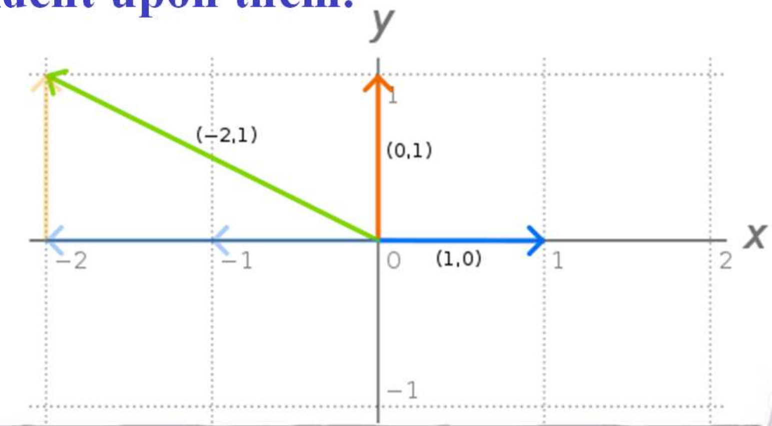
- Given a vector space E and $S \subset E$, the span of S is the subspace of E consisting of all linear combinations of vectors in S.
- Vectors $x_1, \ldots, x_n$ are called linearly independent, if $\sum_{i=1}^{n} \alpha_i x_i = 0$ is true only if $\alpha_i = 0$, for all i. Otherwise, these vectors are linearly dependent.
- A subset $\{x_1, \ldots, x_n\}$ of a vector space E is called a basis for E, when $E = span(x_1, \ldots, x_n)$ and $x_1, \ldots, x_n$ are linearly independent. Then, we say that E has dimension n.

Examples of Bases of Vector Spaces

- In the three-dimensional real vector space R^3 we have the following example:

$$\underbrace{\overbrace{\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\-2\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix}}^{\text{independent}},\begin{bmatrix}4\\2\\3\end{bmatrix}}_{\text{dependent}}$$

- The standard basis in R^2. The blue and orange vectors are the elements of the basis; the green vector can be given in terms of the basis vectors, and so is linearly dependent upon them.



Bases of Hilbert Spaces

✧ A family $\{e_n\}_{n\in\mathbb{N}}$ of a Hilbert space H is orthogonal if for $n \neq p$,

$\langle e_n, e_p \rangle = 0$.

✧ If for $f \in H$ there exists a sequence $a[n]$ such that

$$\lim_{N\to\infty} \left\| f - \sum_{n=0}^{N} a[n] e_n \right\| = 0$$

then $\{e_n\}_{n\in\mathbb{N}}$ is said to be an **orthogonal basis** of H.

✧ The orthogonality implies that necessarily $a[n] = \frac{\langle f, e_n \rangle}{\|e_n\|^2}$ and we write

$$f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n \rangle}{\|e_n\|^2} e_n$$

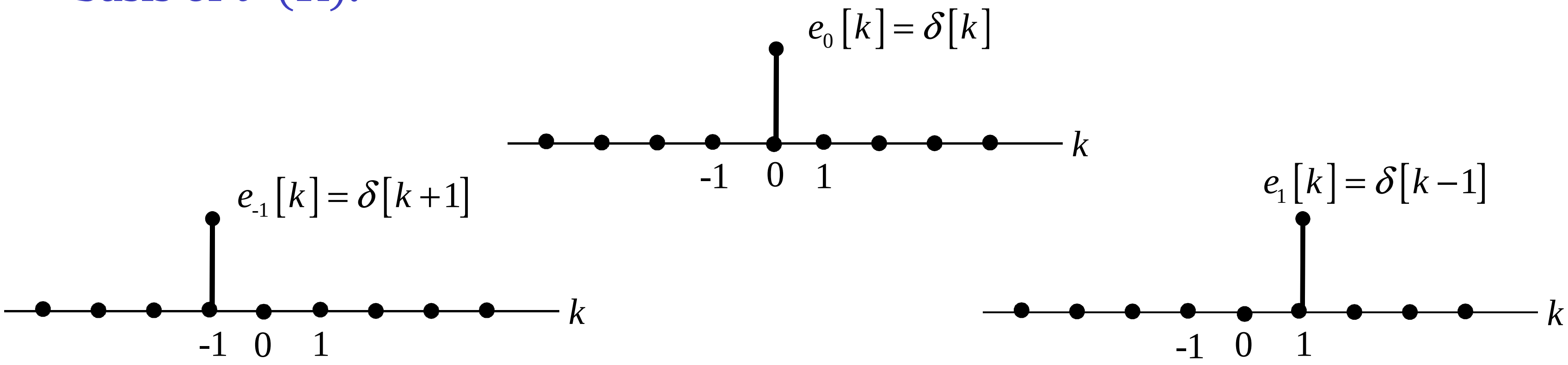
✧ An orthogonal basis is **orthonormal** if $\|e_n\| = 1$ for all $n \in \mathbb{N}$.

Example of Bases of Hilbert Spaces

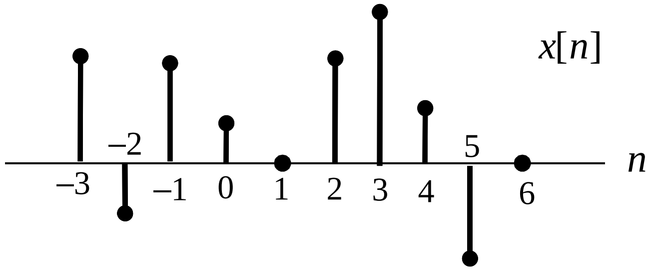
- **Remember that $l^2(\mathbb{R})$ is composed of functions of $\mathbb{Z}$ for which:**

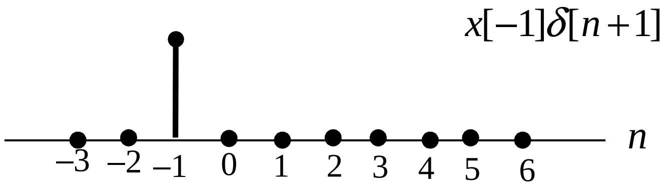
$$\|f\|_2 = \left(\sum_{n=-\infty}^{+\infty} |f[n]|^2 \right)^{1/2} < +\infty$$

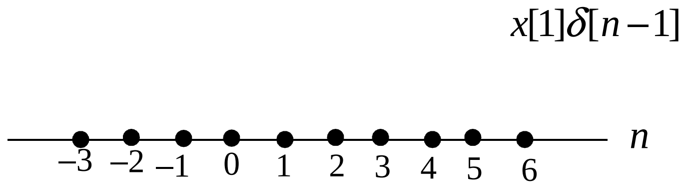
- **The family of translated Diracs $\{e_n[k] = \delta[k-n]\}$ is an orthonormal basis of $l^2(\mathbb{R})$.**

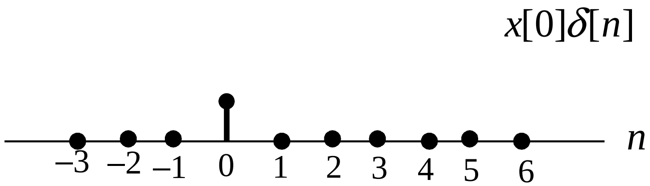


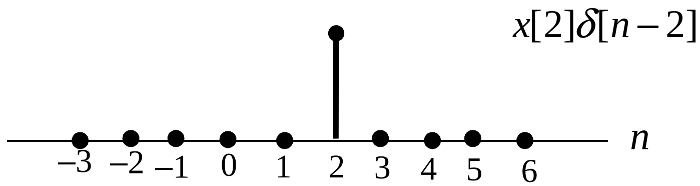
Example of a function in $l^2(\mathbb{R})$ in terms of the family of translated Diracs











$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$\Rightarrow x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Linear Operators

- **Classical signal-processing algorithms are mostly based on linear operators. An operator U from a Hilbert space H_1 to another Hilbert space H_2 is linear if**

$$\forall \lambda_1, \lambda_2 \in \mathbb{C}, \forall f_1, f_2 \in H, U(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 U(f_1) + \lambda_2 U(f_2)$$

- **The null space and image spaces of U are defined by**

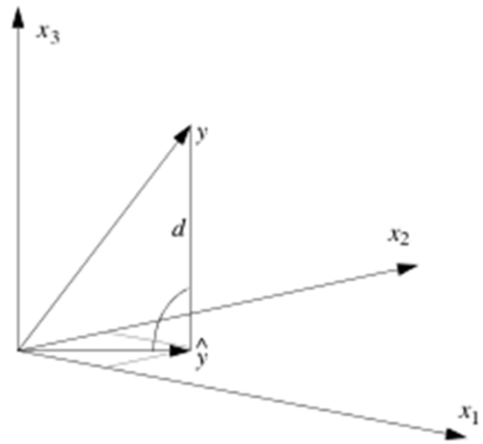
$\boldsymbol{NullU = \{h \in H_1 : Uh = 0\}}$ **and** $\boldsymbol{ImU = \{g \in H_2 : \exists h \in H_1, g = Uh\}}$**.**

Orthogonal Projection and Least-Squares Approximation

- **Often, a vector from a Hilbert space H has to be approximated by a vector lying in a (closed) subspace S.**
- **We assume that S contains an orthonormal basis $\{x_1, x_2, \ldots\}$. Then, the orthogonal projection of $y \in H$ onto S is given by** $\hat{y} = \sum_i \langle x_i, y \rangle x_i$.
- **Note that the difference $d = y - \hat{y}$ satisfies $d \perp S$ and, in particular, $d \perp \hat{y}$.**
- **An important property of such an approximation is that it is best in the least-squares sense, that is** $\min \|y - x\|$ **for x in S is attained for** $x = \sum_i \alpha_i x_i$ **with** $\alpha_i = \langle x_i, y \rangle$.

Example of Orthogonal Projection

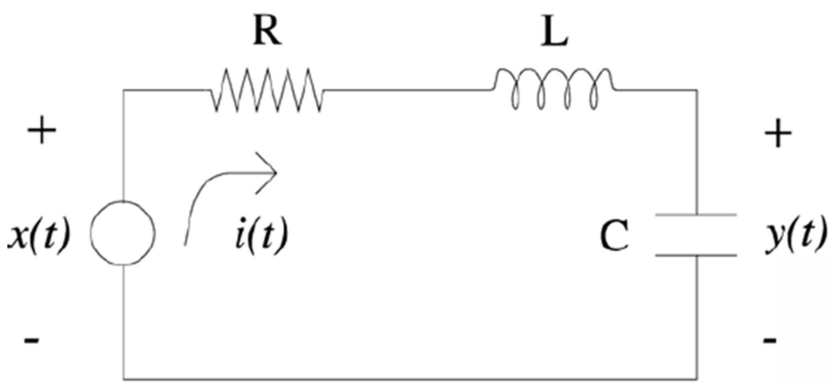
- **Here, $y \in \mathbb{R}^3$ and $\hat{y}$ is its projection onto the span of $\{x_1, x_2\}$. Note that $y - \hat{y}$ is orthogonal to the span $\{x_1, x_2\}$.**



SYSTEM EXAMPLES

$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t) \qquad x[t] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[t]$$

✧ Example 1: RLC circute

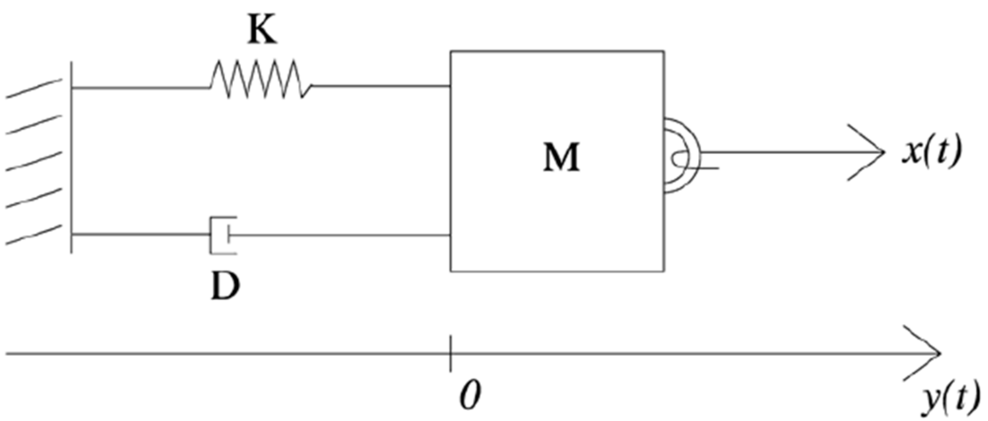


$$R \times i(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(y)}{dt}$$

$$\longrightarrow \quad LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Example 2: Mechanical system



$x(t)$ - applied force

K - spring constant

D - damping constant

$y(t)$ - displacement from rest

$$\mathrm{M}\frac{d^2y(t)}{dt^2} = x(t) - \mathrm{K}y(t) - \mathrm{D}\frac{dy(t)}{dt}$$

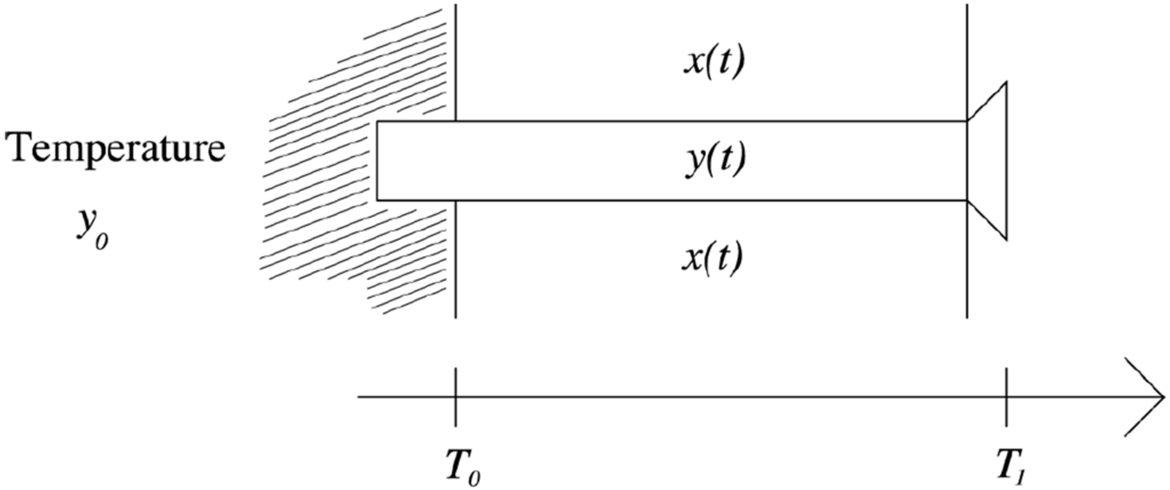
$$\Downarrow$$

$$\mathrm{M}\frac{d^2y(t)}{dt^2} + \mathrm{D}\frac{dy(t)}{dt} + \mathrm{K}y(t) = x(t)$$

✧ **Observation: Very different physical systems may be modeled mathematically in very similar ways.**

Example 3: Thermal System

- Cooling Fin in Steady State



t = distance along rode

y(t) = Fin temperature as function of position

x(t) = Surrounding temperature along the fin

Example 3 (Continued)

$$\frac{d^2y(t)}{dt^2} = k[y(t) - x(t)]$$

$$y(T_0) = y_0$$

$$\frac{dy}{dt}(T_1) = 0$$

Observations

- ✧ **Independent variable can be something other than time, such as space.**
- ✧ **Such systems may, more naturally, have boundary conditions, rather than "initial" conditions**

Example 4: Financial system

- **Fluctuations in the price of zero-coupon bonds**

  t = 0 Time of purchase at price y_0

  t = T Time of maturity at value y_T

  y(t) = Values of bond at time t

  x(t) = Influence of external factors on fluctuations in bond price

$$\frac{d^2y(t)}{dt^2} = f\left(y(t), \frac{dy(t)}{dt}, x_1(t), x_2(t), \ldots, x_N(t), t\right)$$

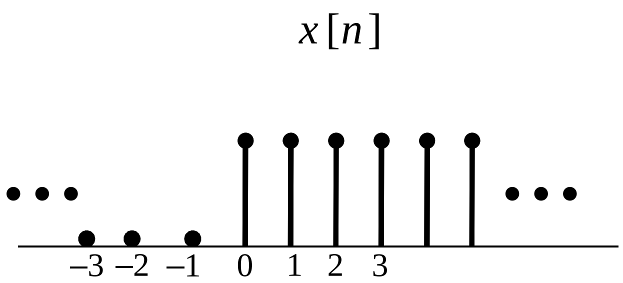
$$y(0) = y_0, \quad y(T) = y_T.$$

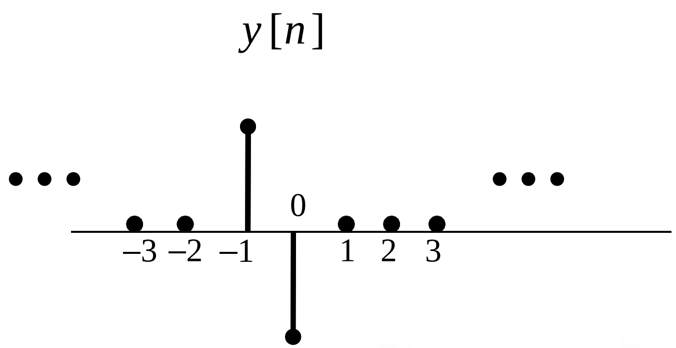
- **Observation: Even if the independent variable is time, there are interesting and important systems which have boundary conditions.**

Example 5: A rudimentary "edge" detector

$$y[n] = x[n+1] - 2x[n] + x[n-1]$$
$$= \{x[n+1] - x[n]\} - \{x[n] - x[n-1]\}$$

✧ This system detects changes in signal slope





Example 5 (continued)

- **Observations:**
  - A very rich class of systems (but by no means all systems of interest to us) are described by differential and difference equations.
  - Such an equation, by itself, does not completely describe the input-output behavior of a system: we need auxiliary conditions (initial conditions, boundary conditions).
  - In some cases the system of interest has time as the natural independent variable and is causal. However, that is not always the case.
  - Very different physical systems may have very similar mathematical descriptions.

SYSTEM PROPERTIES
(Causality, Linearity, Time-invariance, etc.)

WHY?

- ✧ **Important practical/physical implications**

- ✧ **They provide us with insight and structure that we can exploit both to analyze and understand systems more deeply.**

CAUSALITY

- A system is causal if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.
- All real-time physical systems are causal, because time only moves forward.
- Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)
- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)
- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

CAUSALITY (continued)

- **Mathematically (in CT): A system x(t) →y(t) is causal if**

when $x_1(t) \to y_1(t)$ **,** $x_2(t) \to y_2(t)$

and $x_1(t) = x_2(t)$ **for all** $t \le t_0$

Then $y_1(t) = y_2(t)$ **for all** $t \le t_0$

CAUSAL OR NONCAUSAL

$$y(t) = x^2(t-1)$$

- ✧ **Causal, e.g. y(5) depends on x(4).**

$$y(t) = x(-t)$$

- ✧ **Noncausal, e.g. y(5) = x(-5) ok, but y(-5) = x(5), y depends on future.**

$$y(t) = 0.5^{n+1} x^3(t-1)$$

- ✧ **Causal, e.g. y(5) depends on x(4).**

TIME-INVARIANCE (TI)

- **Informally, a system is time-invariant (TI) if its behavior does not depend on what time it is.**

- **Mathematically (in DT): A system x[n] → y[n] is TI if for any input x[n] and anytime shift n_0,**

  If $x[n] \to y[n]$ **then** $x[n-n_0] \to y[n-n_0]$.

- **Similarly for a CT time-invariant system,**

  If $x(n) \to y(n)$ **then** $x(n-n_0) \to y(n-n_0)$.

TIME-INVARIANT OR TIME-VARYING ?

$$y(t) = x^2(t+1)$$

- TI.

$$y[n] = 0.5^{n+1} x^3[n-1]$$

- Not TI.

NOW WE CAN DEDUCE SOMETHING!

- **Fact: If the input to a TI System is periodic, then the output is periodic with the same period.**
- **"Proof":**

Suppose $x(t) = x(t + T)$ and $x(t) \rightarrow y(t)$,

by TI $\quad x(t + T) \rightarrow y(t + T)$

It is same as $x(t)$!

So $y(t + T)$ is the same as $y(t)$, i.e.,

$y(t) = y(t + T)$.

LINEARITY

- A (CT) system is linear if it has the superposition property:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

$$y(t) = x^2(t)$$

- Nonlinear, TI, Causal.

$$y(t) = x(2t)$$

- Linear, **not** TI, Noncausal.

- Can you find systems with other combinations ? -e.g.:
  - Linear, TI, Noncausal
  - Linear, not TI, Causal

LINEAR AND NONLINEAR SYSTEMS

- Many systems are nonlinear. For example: many circuit elements (e.g., diodes), dynamics of aircraft, econometric models,…
- However we focus exclusively on linear systems.
- Why?
  - Linear models represent accurate representations of behavior of many systems (e.g., linear resistors, capacitors, other examples given previously,…)
  - Can often linearize models to examine “small signal” perturbations around “operating points”
  - Linear systems are analytically tractable, providing basis for important tools and considerable insight

PROPERTIES OF LINEAR SYSTEMS

- **Superposition**

  If $x_k[n] \rightarrow y_k[n]$

  Then $\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$

- **For linear systems, zero input → zero output**

  “Proof”: $0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0$

Properties of Linear Systems (continued)

- **A linear system is causal if and only if it satisfies the condition of initial rest:**

$$x(t)=0 \ \textbf{for} \ t \le t_0 \rightarrow x(t)=0 \ \textbf{for} \ t \le t_0$$

LINEAR TIME-INVARIANT (LTI) SYSTEMS

- **Focus of most of this course**
  - **Practical importance (Example 1-3 earlier this lecture are all LTI systems.)**
  - **The powerful analysis tools associated with LTI systems**

- **A basic fact: If we know the response of an LTI system to some inputs, we actually know the response to many inputs**