Digital Media Laboratory
Advanced Information & Communication Technology Center
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Signals & Systems

LTI Systems (Impulse Response & Convolution)

Adapted from: Lecture notes from MIT

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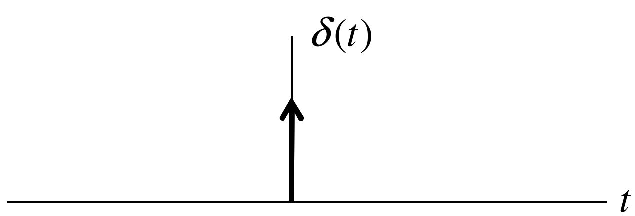
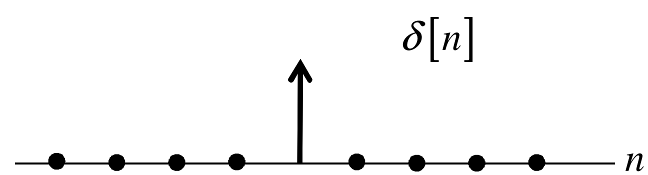
CT & DT Signals in terms of shifted impulses

$$x[n]=\sum_k a_k x_k[n] \xrightarrow{LinearSystem} y[n]=\sum_k a_k y_k[n]$$

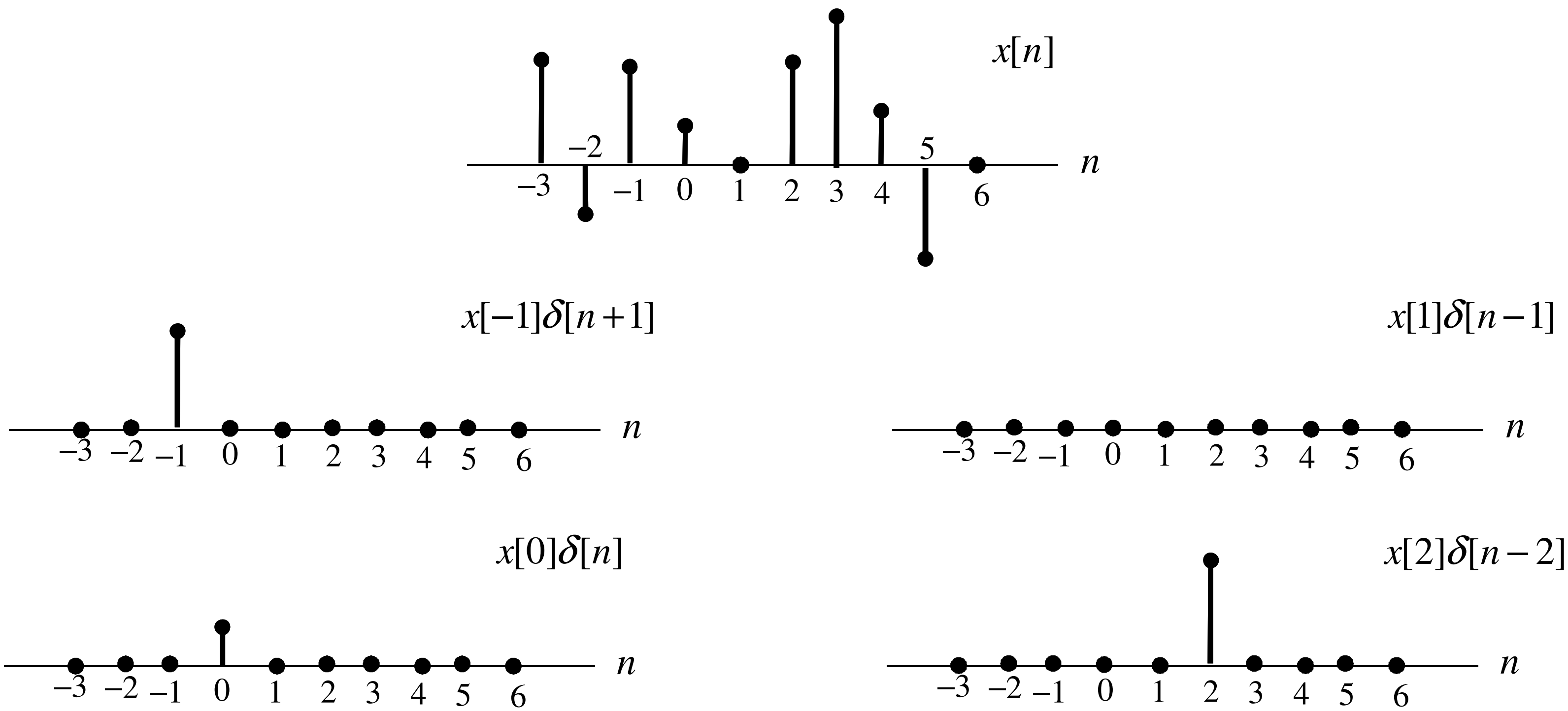
Question: Are there sets of "basic" signals so that:

- **a) We can represent rich classes of signals as linear combinations of these building block signals.**
- **b) The response of LTI systems to these basic signals are both simple and insightful.**





DT Signals in terms of shifted unit samples



$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$\Rightarrow x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

DT Unit Sample Response

$$\delta[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow h[n]$$

Suppose the system is linear and define $h_k[n]$ as the response to $\delta[n-k]$.

$$\delta[n-k] \;\rightarrow\; h_k[n]$$

From Superposition:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow y[n] = \sum_{k\rightarrow-\infty}^{\infty} x[k]h_k[n]$$

DT Unit Sample Response

$$\delta[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow h[n]$$

Now Suppose the system is LTI and define $h[n]$ as the unit sample response.

$$\delta[n] \;\rightarrow\; h[n]$$

From TI:

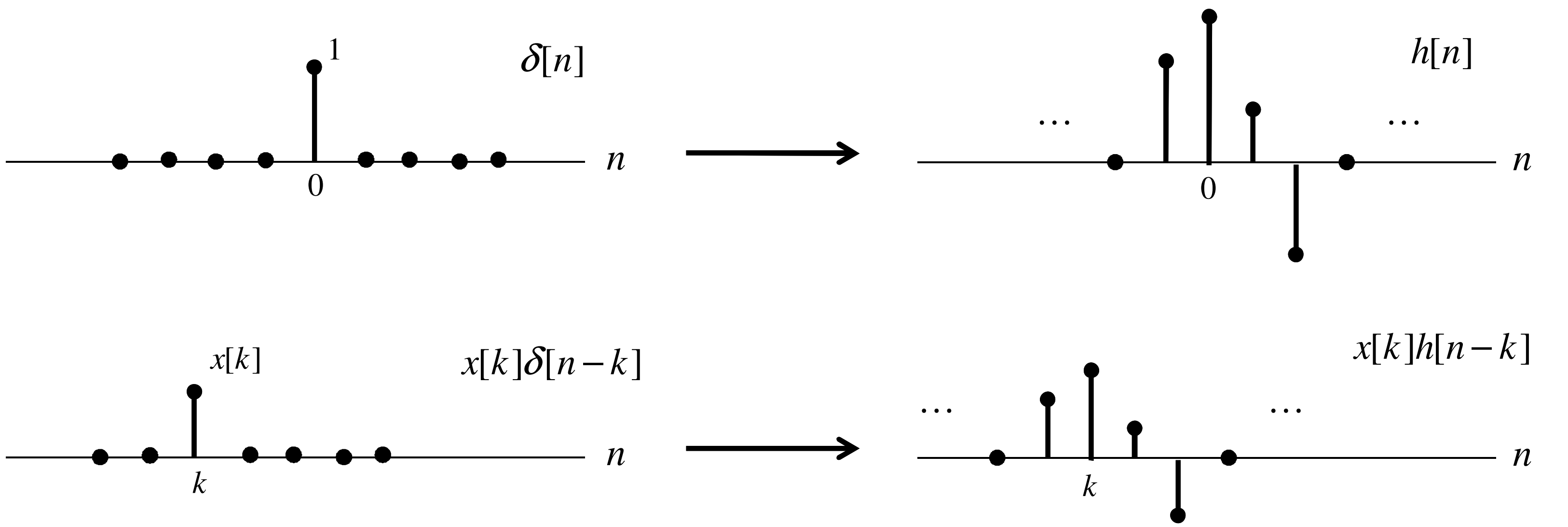
$$\delta[n-k] \;\rightarrow\; h[n-k]$$

From LTI:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \underbrace{y[n] = \sum_{k\rightarrow-\infty}^{\infty} x[k]h[n-k]}_{\text{Convolution Sum}}$$

Convolution Sum Representation of Response of LTI Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



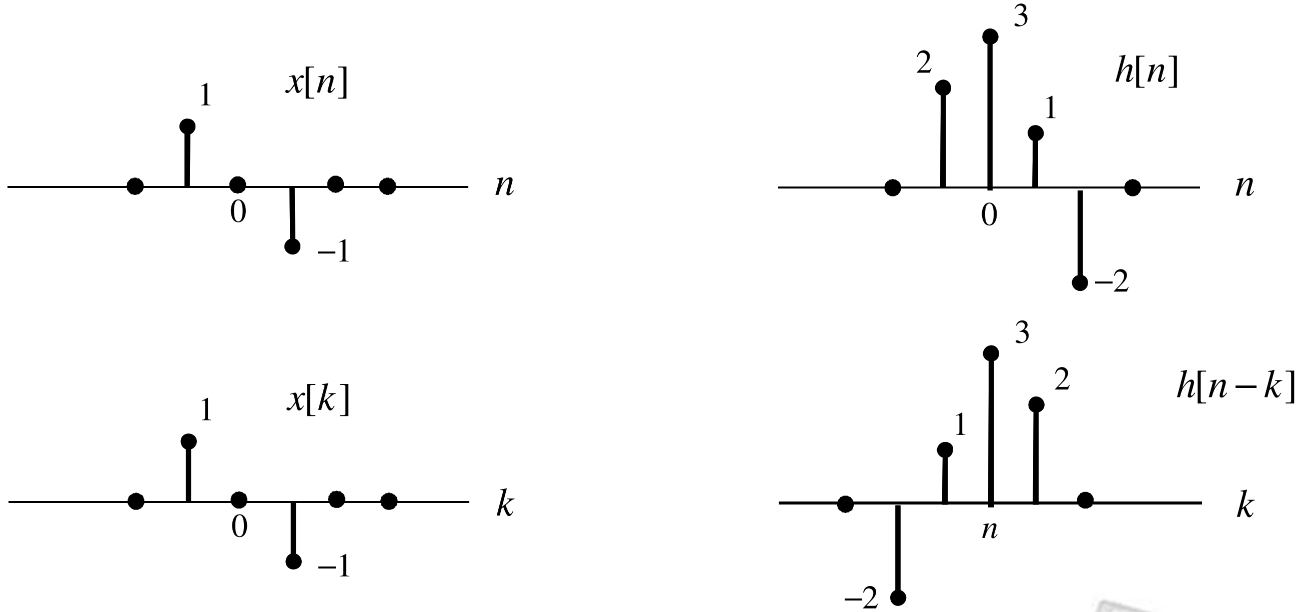
Sum up responses over all k

Visualizing the convolution sum

Choose a value for n and consider it fixed.

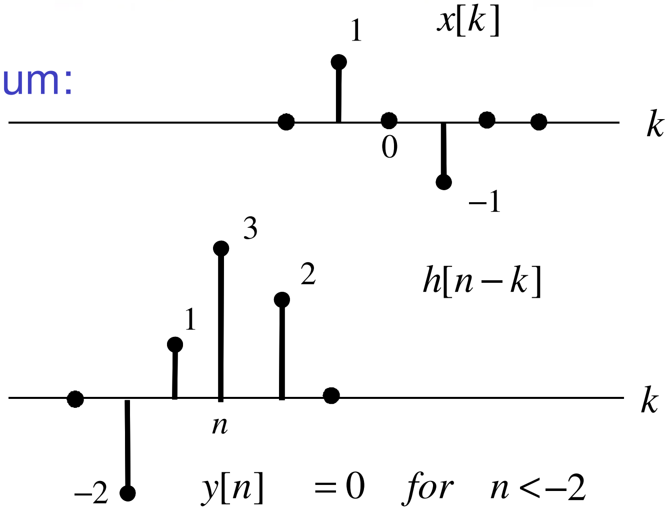
$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

View as function of k with n fixed



Visualizing the convolution sum

Shift, Multiply, and Sum:



$$
\begin{aligned}
y[n] &= 0 \quad for \quad n < -2 \\
y[-2] &= \quad 0\times 3 + 1\times 2 = 2 \\
y[-1] &= \quad 0\times 1 + 1\times 3 + 0\times 2 = 3 \\
y[0] &= \quad 0\times(-2) + 1\times 1 + 0\times 3 + (-1)\times 2 = -1 \\
y[1] &= \quad 1\times(-2) + 0\times 1 + (-1)\times 3 = -5 \\
y[2] &= \quad 0\times(-2) + (-1)\times 1 = -1 \\
y[3] &= \quad (-1)\times(-2) = 2 \\
y[4] &= \quad 0 \\
y[n] &= 0 \quad for \quad n > 3
\end{aligned}
$$

Properties of Convolution and DT LTI systems

1. A DT LTI system is **completely characterized** by its unit sample response:

   Ex. 1. $h[n] = \delta[n - n_0]$

   There are **many systems** with this response to $\delta[n]$.

   There is **only one** LTI system with this response to $\delta[n]$:

$$y[n] = x[n - n_0]$$

$$\Downarrow$$

$$x[n] * \delta[n - n_0] = x[n - n_0]$$

Properties of Convolution and DT LTI systems

Ex. 2 $y[n] = \sum_{k=-\infty}^{n} x[k]$ **- An Accumulator**

Unit sample response:

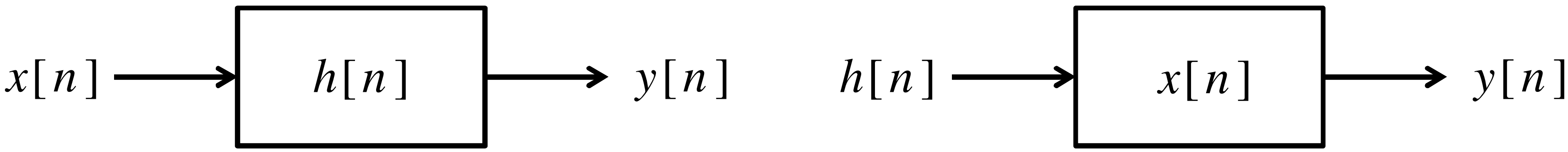
$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

$$\Downarrow$$

$$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$$

The Commutative Property

$$y[n] = x[n] * h[n] = h[n] * x[n]$$



Ex. Step response $s[n]$ **of an LTI System:**

$$s[n] = u[n] * h[n] = h[n] * u[n]$$

Step Input

"Input"

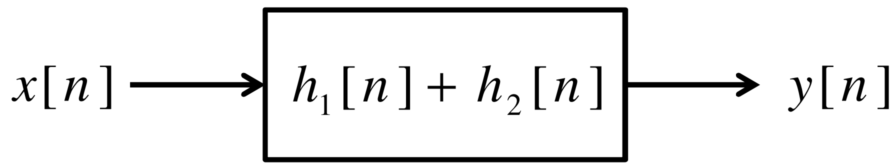
Unit Sample response of accumulator

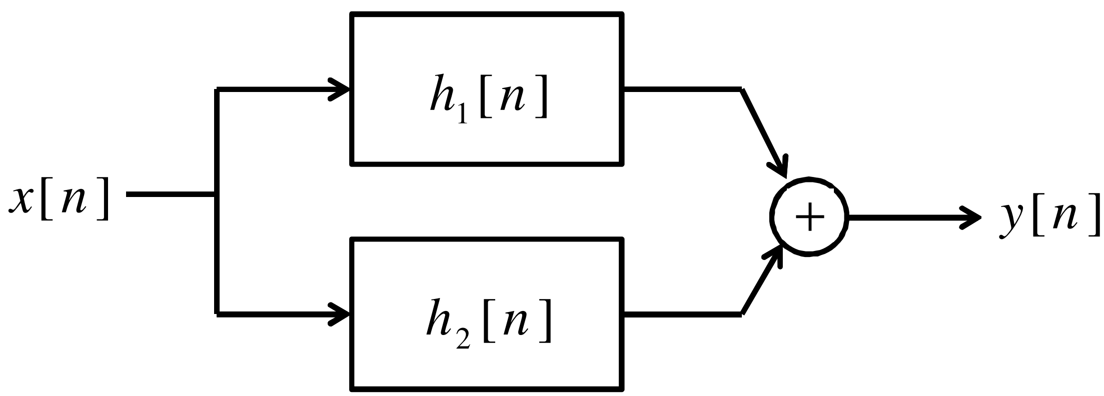
$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

The Distributive Property

$$x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$$

Interpretation:



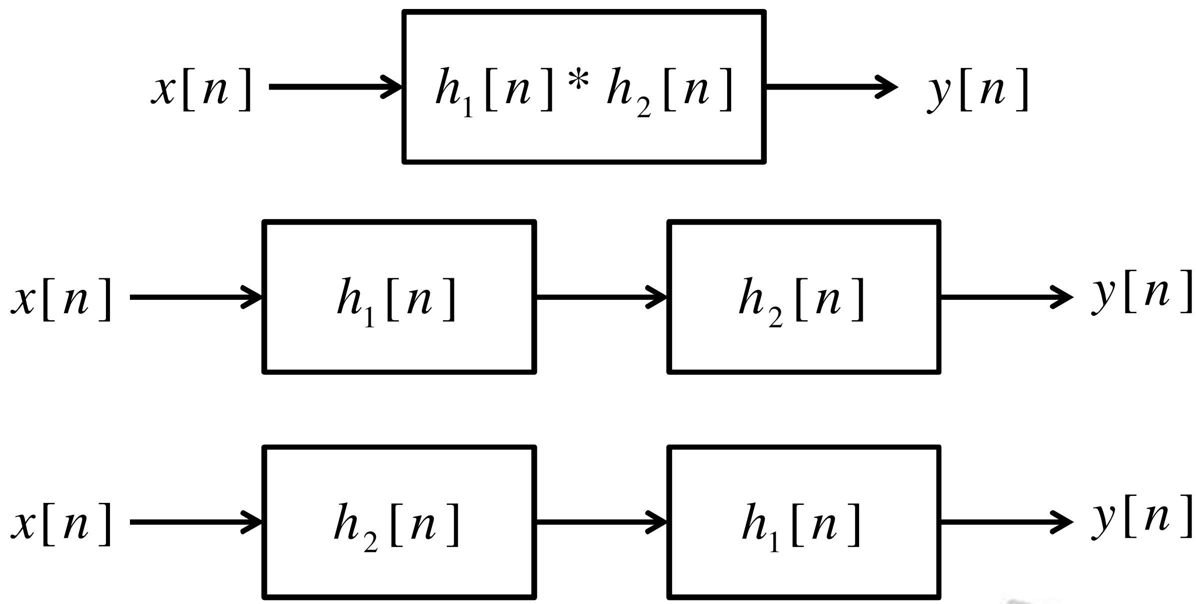


The Associative Property

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

Commutativity:

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$



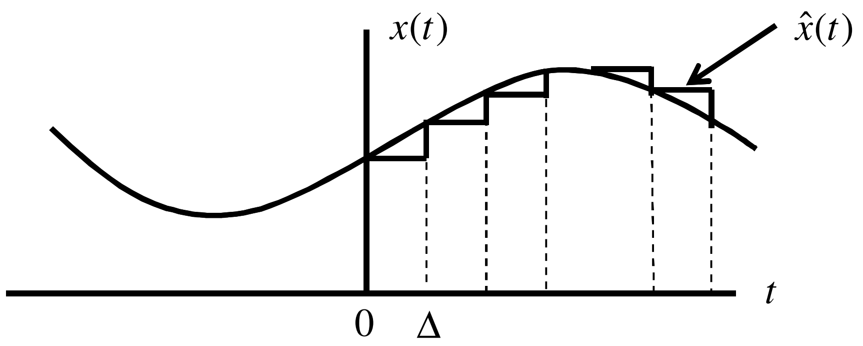
Properties of LTI Systems

1. **Causality** $\Longleftrightarrow$ $h[n] = 0 \quad for\ all \ \ n < 0$

2. **Stability** $\Longleftrightarrow$ $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

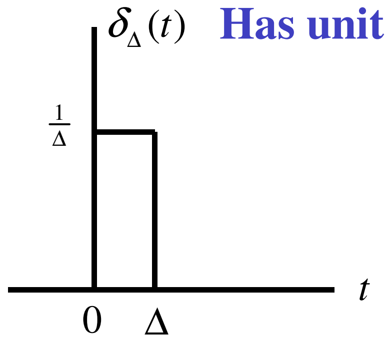
Representation of CT Signals

Approximate any input $x(t)$ as a sum of shifted, scaled pulses:

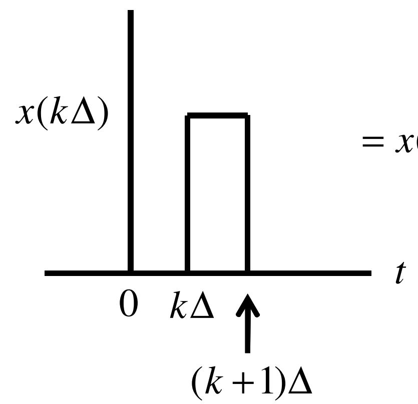


$$\hat{x}(t) = x(k\Delta),\ k\Delta < t < (k+1)\Delta$$

Representation of CT Signals



Has unit area



$$= x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\Downarrow \ \textit{limit as } \Delta \to 0$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

CT Impulse Response

$$\delta(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow h(t)$$

$$\delta(t) \;\rightarrow\; h(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \underbrace{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{\text{Convolution Integral}}$$