



Digital Media Laboratory
Advanced Information & Communication Technology Center
Sharif University of Technology

Signals & Systems

LTI Systems (Impulse Response & Convolution)

Adapted from: Lecture notes from MIT

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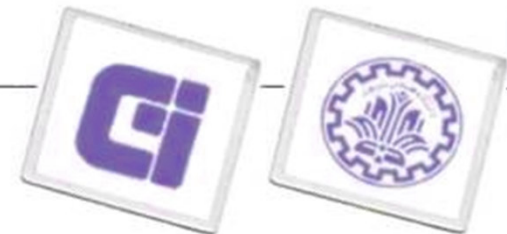
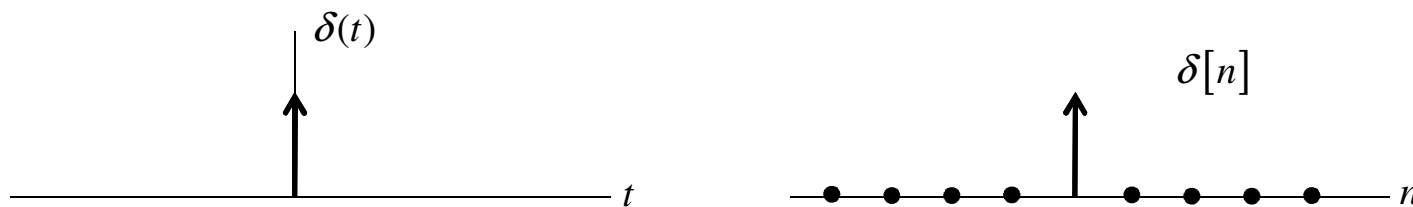
Fall 2012

CT & DT Signals in terms of shifted impulses

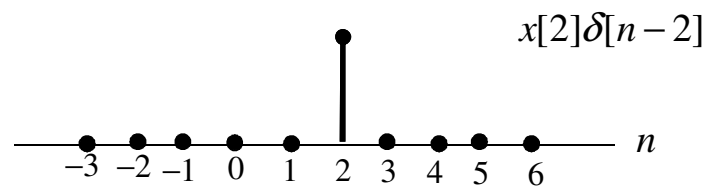
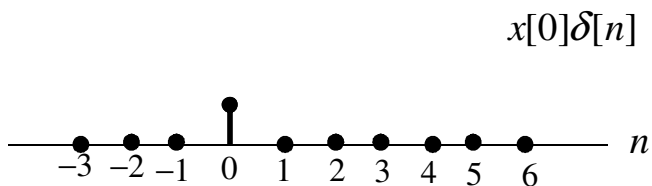
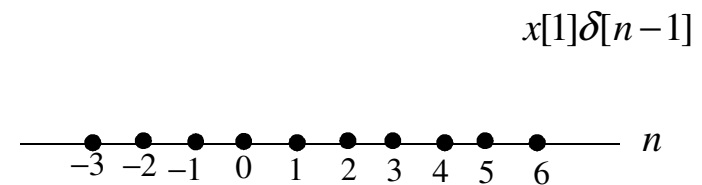
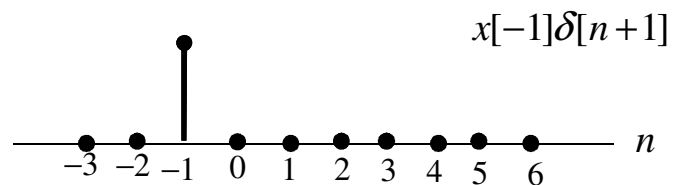
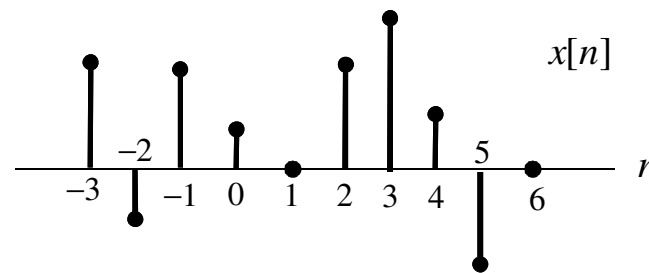
$$x[n] = \sum_k a_k x_k[n] \xrightarrow{\text{LinearSystem}} y[n] = \sum_k a_k y_k[n]$$

Question: Are there sets of “basic” signals so that:

- ✧ a) We can represent rich classes of signals as **linear combinations** of these **building block signals**.
- ✧ b) The response of LTI systems to these basic signals are both **simple and insightful**.

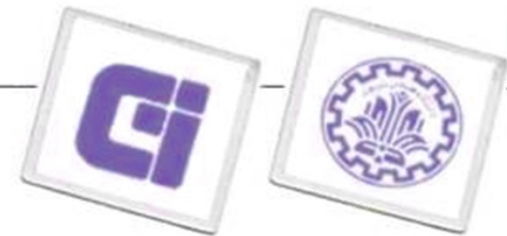


DT Signals in terms of shifted unit samples

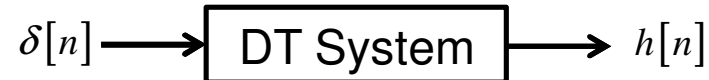


$$x[n] = \dots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$\Rightarrow x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$



DT Unit Sample Response

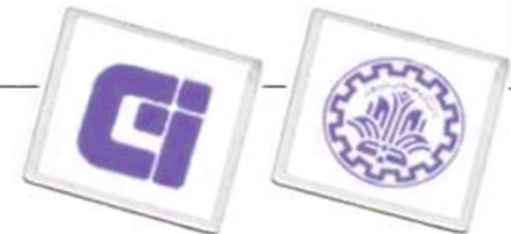


Suppose the system is **linear** and define $h_k[n]$ as the response to $\delta[n-k]$.

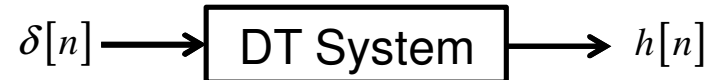
$$\delta[n-k] \rightarrow h_k[n]$$

From Superposition:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$



DT Unit Sample Response



Now Suppose the system is **LTI** and define $h[n]$ as the unit sample response.

$$\delta[n] \rightarrow h[n]$$

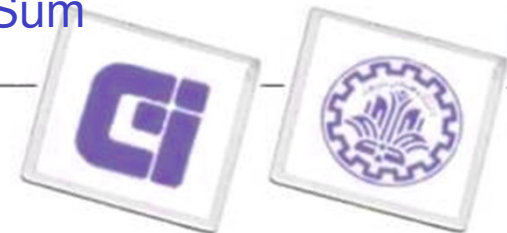
From TI:

$$\delta[n-k] \rightarrow h[n-k]$$

From LTI:

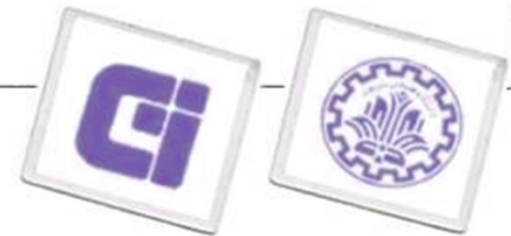
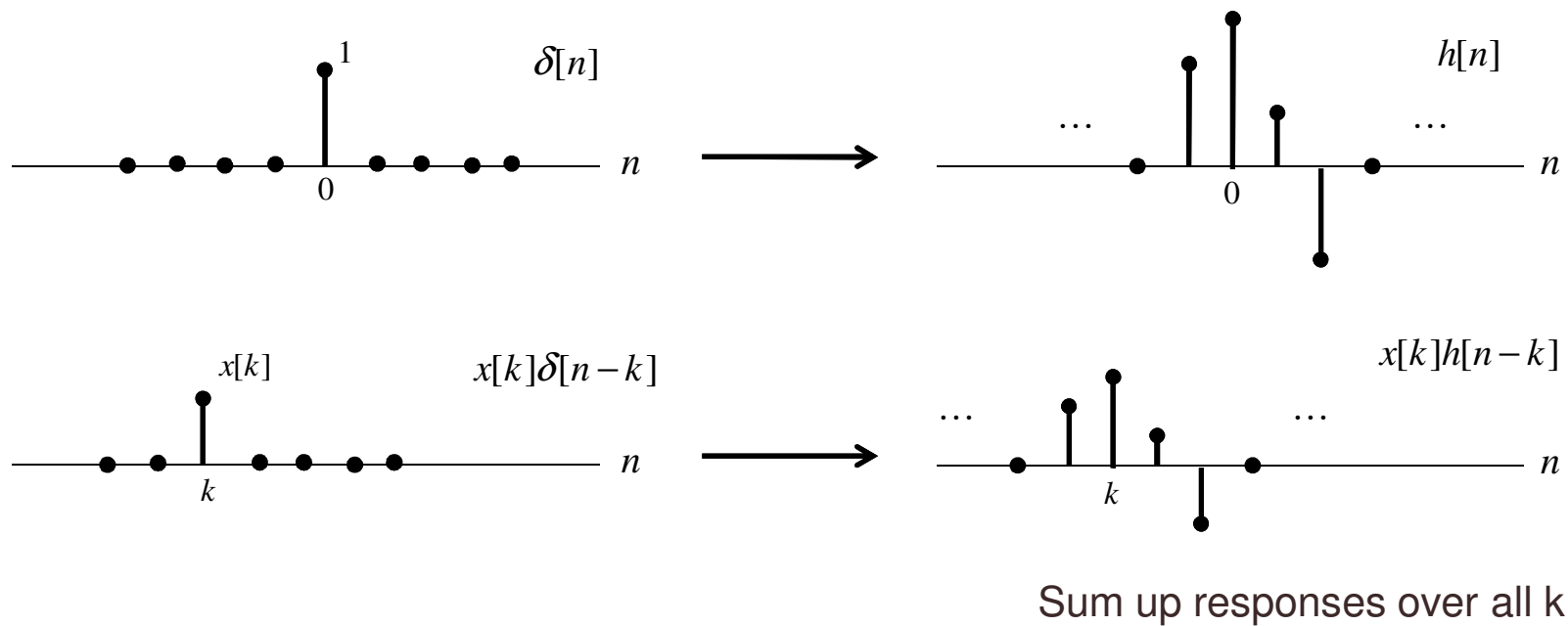
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow y[n] = \underbrace{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{\text{Convolution Sum}}$$

Convolution Sum



Convolution Sum Representation of Response of LTI Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

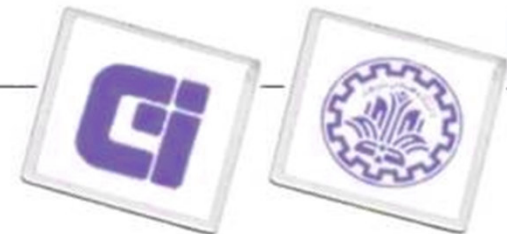
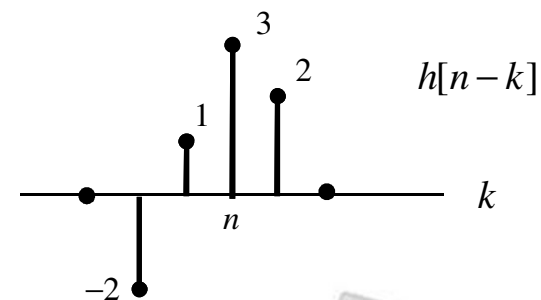
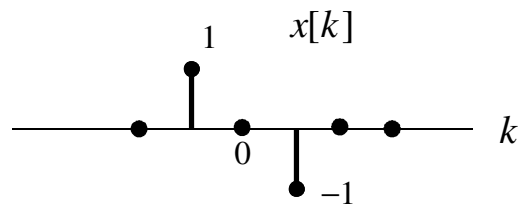
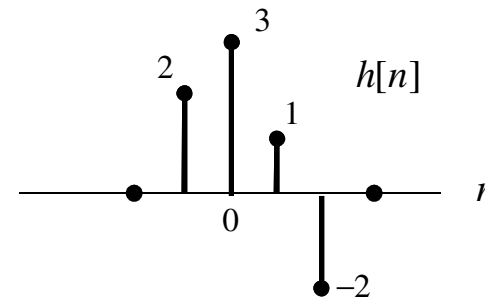
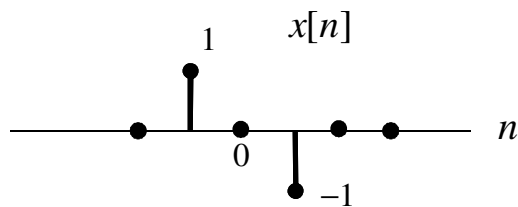


Visualizing the convolution sum

Choose a value for n and consider it fixed.

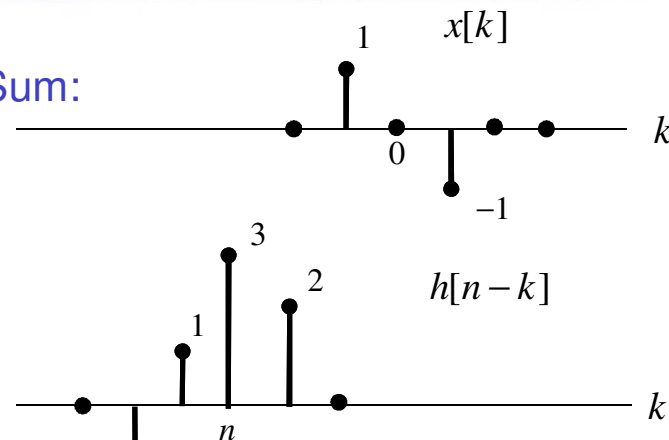
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

View as function of k with n fixed



Visualizing the convolution sum

Shift, Multiply, and Sum:



$$y[n] = 0 \quad \text{for } n < -2$$

$$y[-2] = 0 \times 3 + 1 \times 2 = 2$$

$$y[-1] = 0 \times 1 + 1 \times 3 + 0 \times 2 = 3$$

$$y[0] = 0 \times (-2) + 1 \times 1 + 0 \times 3 + (-1) \times 2 = -1$$

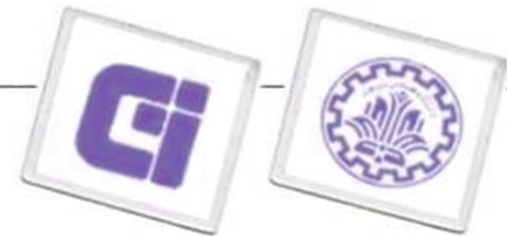
$$y[1] = 1 \times (-2) + 0 \times 1 + (-1) \times 3 = -5$$

$$y[2] = 0 \times (-2) + (-1) \times 1 = -1$$

$$y[3] = (-1) \times (-2) = 2$$

$$y[4] = 0$$

$$y[n] = 0 \quad \text{for } n > 3$$



Properties of Convolution and DT LTI systems

1. A DT LTI system is completely characterized by its unit sample response:

Ex. 1. $h[n] = \delta[n - n_0]$

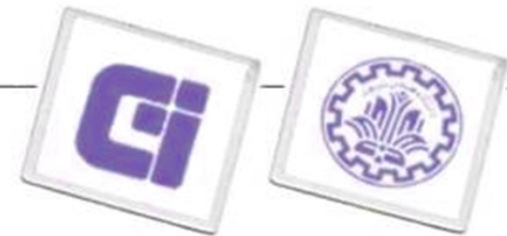
There are many systems with this response to $\delta[n]$.

There is only one LTI system with this response to $\delta[n]$:

$$y[n] = x[n - n_0]$$



$$x[n] * \delta[n - n_0] = x[n - n_0]$$



Properties of Convolution and DT LTI systems

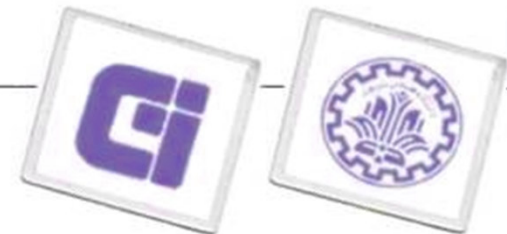
Ex. 2 $y[n] = \sum_{k=-\infty}^n x[k]$ - An Accumulator

Unit sample response:

$$h[n] = \sum_{k=-\infty}^n \delta[k] = u[n]$$

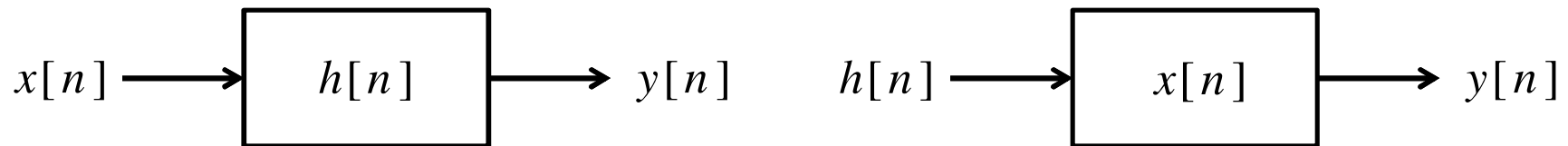


$$x[n] * u[n] = \sum_{k=-\infty}^n x[k]$$



The Commutative Property

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

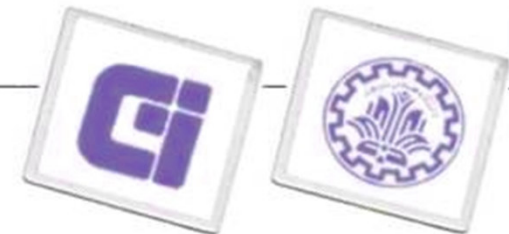


Ex. Step response $s[n]$ of an LTI System:

$$s[n] = u[n] * h[n] = h[n] * u[n]$$

\uparrow Step Input \uparrow "Input" \uparrow Unit Sample response of accumulator

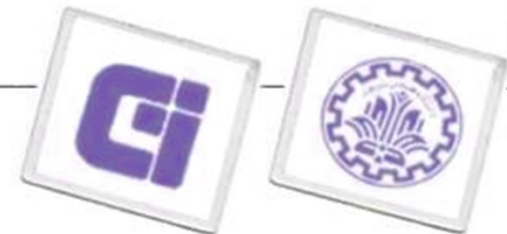
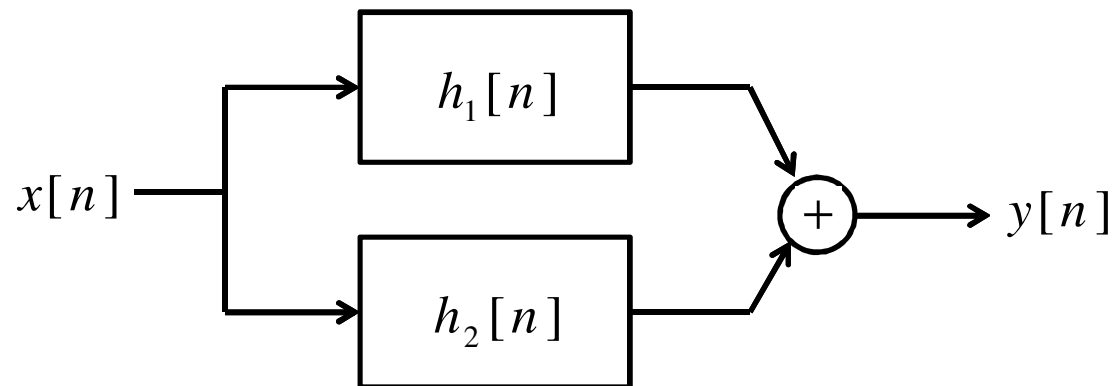
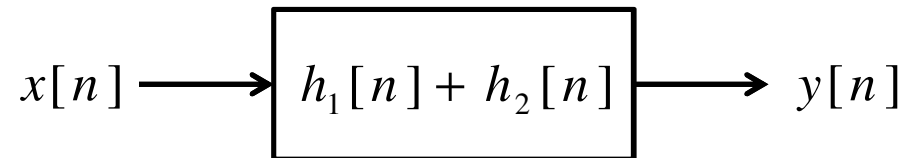
$$s[n] = \sum_{k=-\infty}^n h[k]$$



The Distributive Property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

Interpretation:

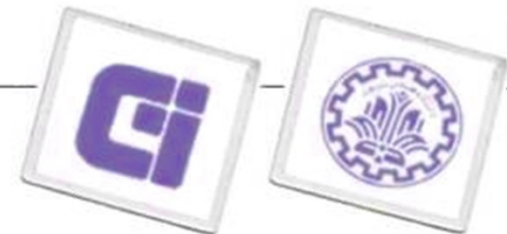
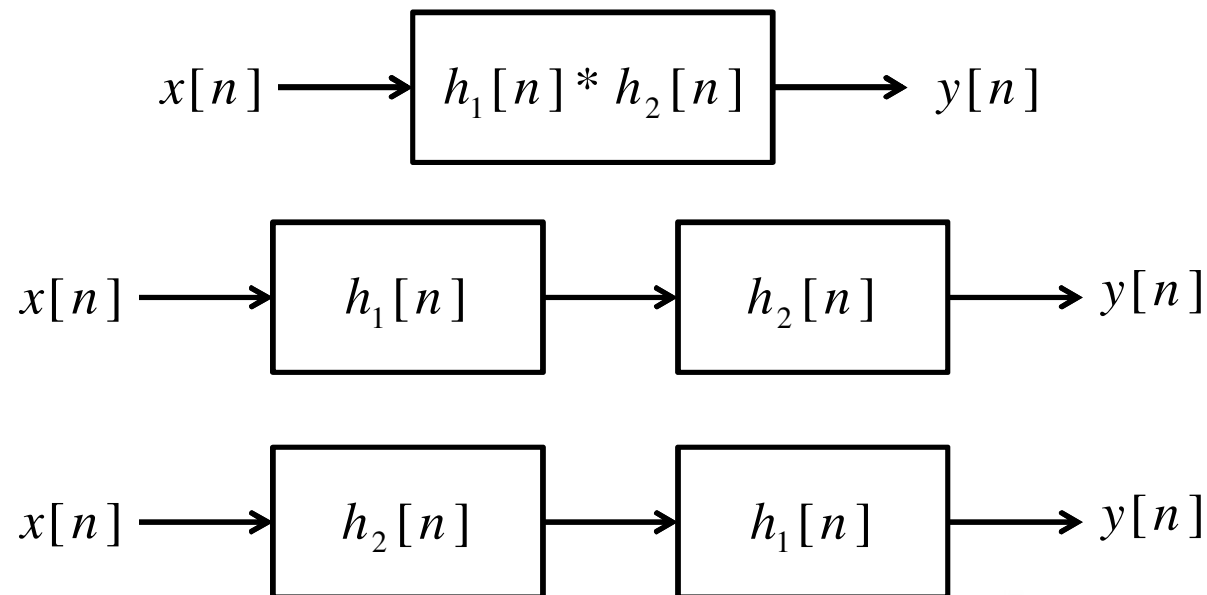


The Associative Property

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

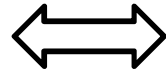
Commutativity:

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$



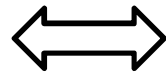
Properties of LTI Systems

1. Causality

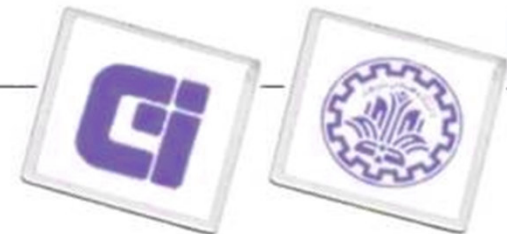


$$h[n] = 0 \text{ for all } n < 0$$

2. Stability

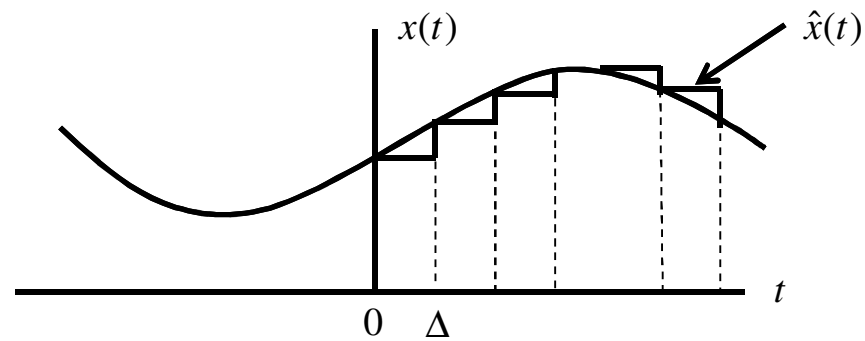


$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$

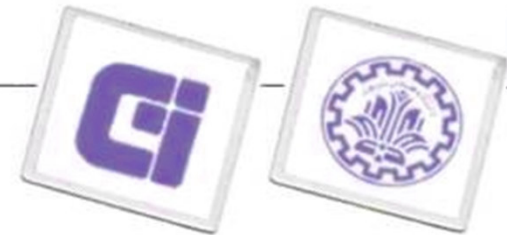


Representation of CT Signals

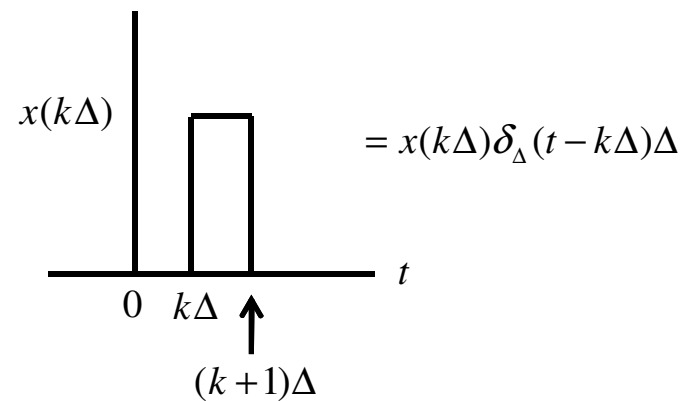
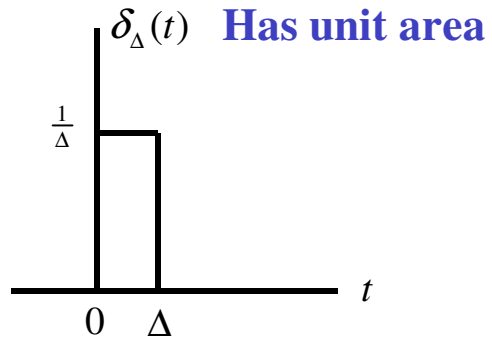
Approximate any input $x(t)$ as a sum of shifted, scaled pulses:



$$\hat{x}(t) = x(k\Delta), k\Delta < t < (k+1)\Delta$$



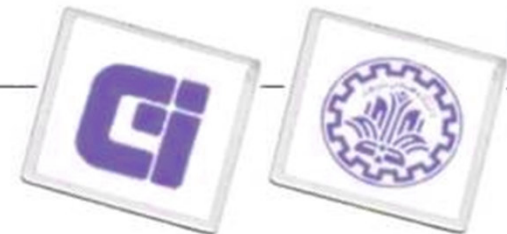
Representation of CT Signals



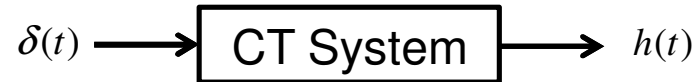
$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

↓ limit as $\Delta \rightarrow 0$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$



CT Impulse Response



$$\delta(t) \rightarrow h(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow y(t) = \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{\text{Convolution Integral}}$$

Convolution Integral

