



Digital Media Laboratory
Advanced Information & Communication Technology Center
Sharif University of Technology

Signals & Systems

LTI Systems (CT Convolution & Difference Equations)

Adapted from: Lecture notes from MIT

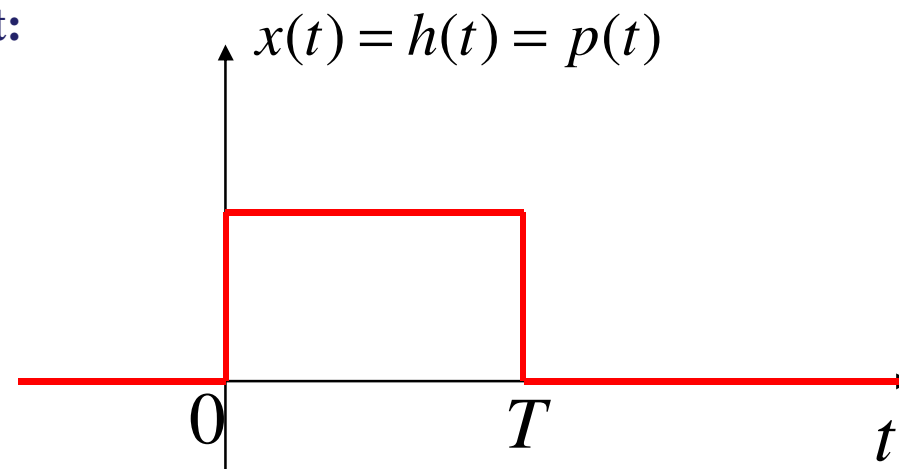
Dr. Hamid R. Rabiee

Fall 2012

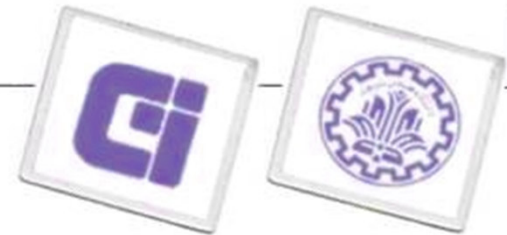
Example of CT convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Suppose that:

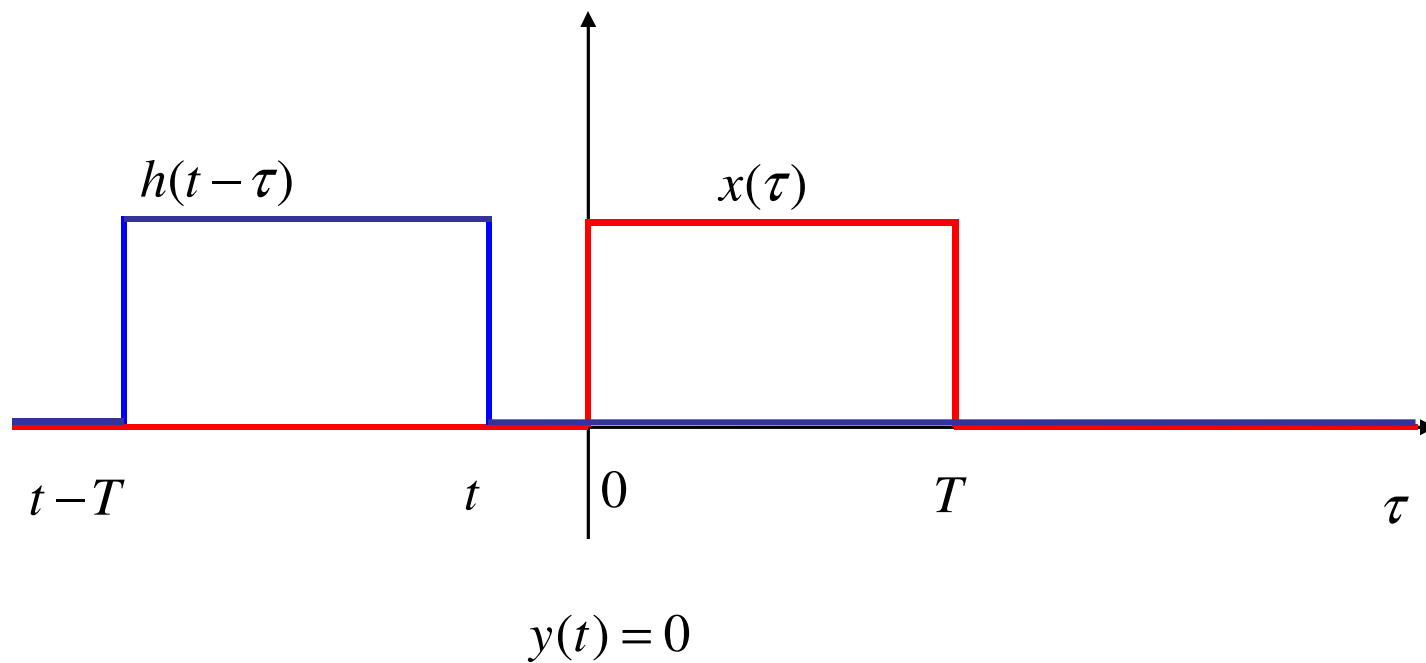


We have to consider four cases.



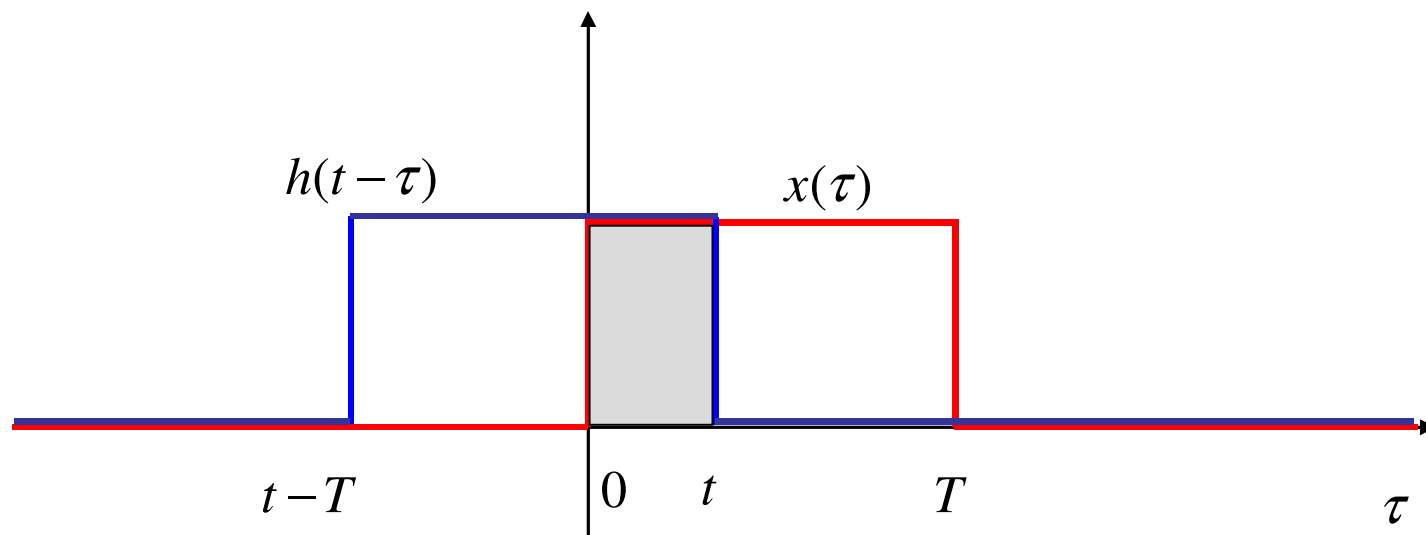
Example of CT convolution

✧ Case 1: $t \leq 0$

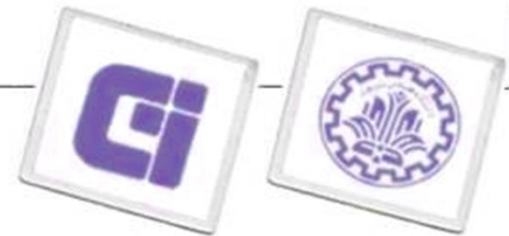


Example of CT convolution

✧ Case 2: $0 \leq t \leq T$

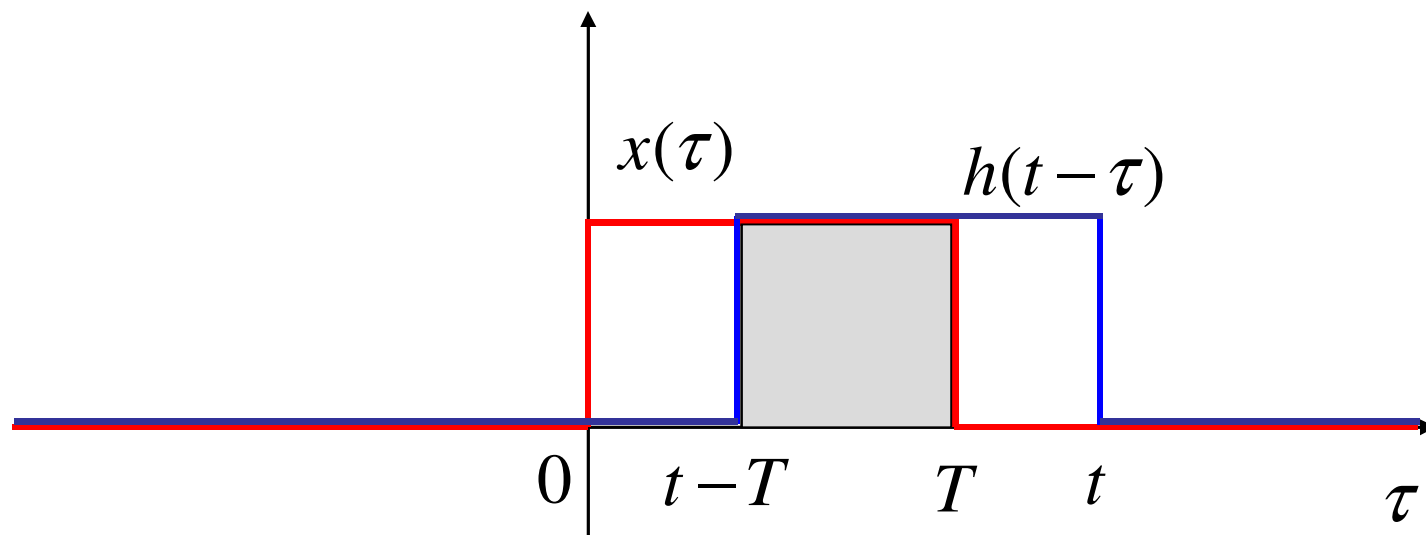


$$y(t) = \int_0^t d\tau = t$$

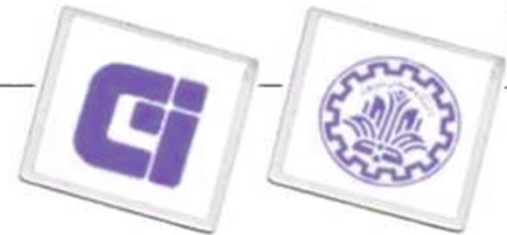


Example of CT convolution

✧ Case 3: $0 \leq t - T \leq T \rightarrow T \leq t \leq 2T$



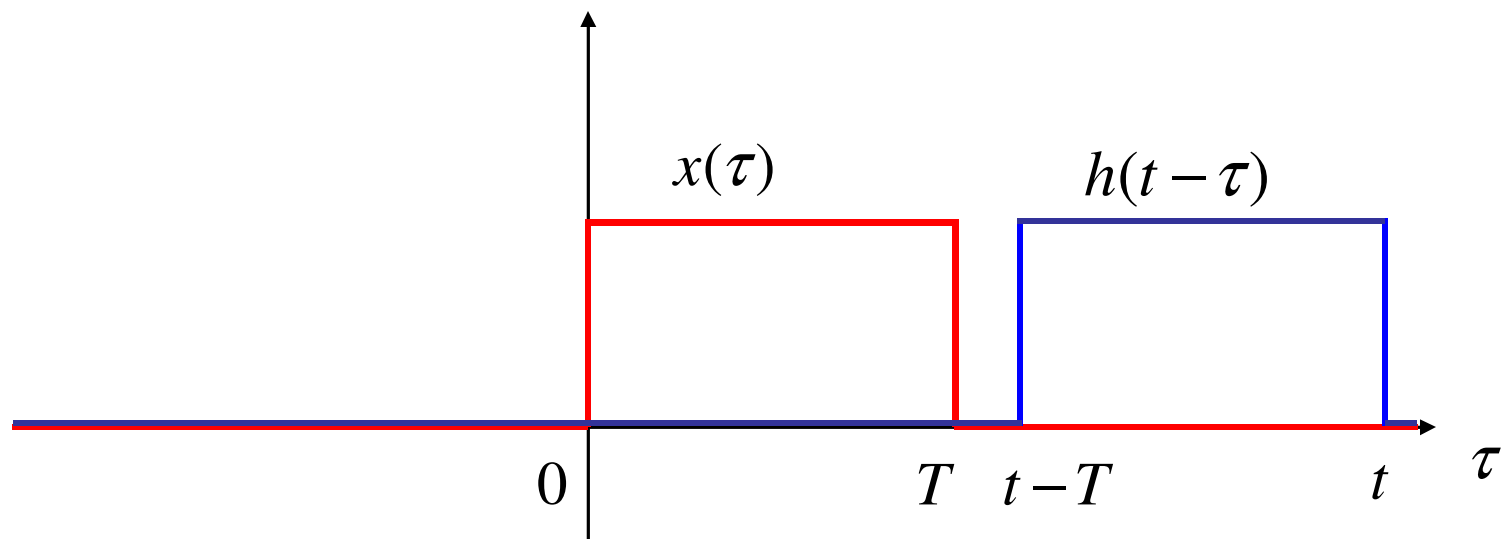
$$y(t) = \int_{t-T}^T d\tau = T - (t - T) = 2T - t$$



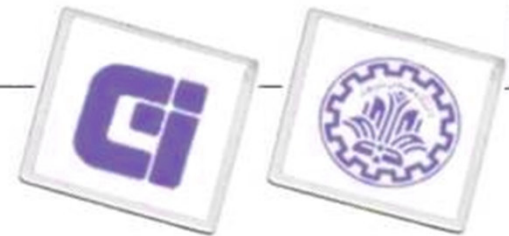
Example of CT convolution

✧ Case 4:

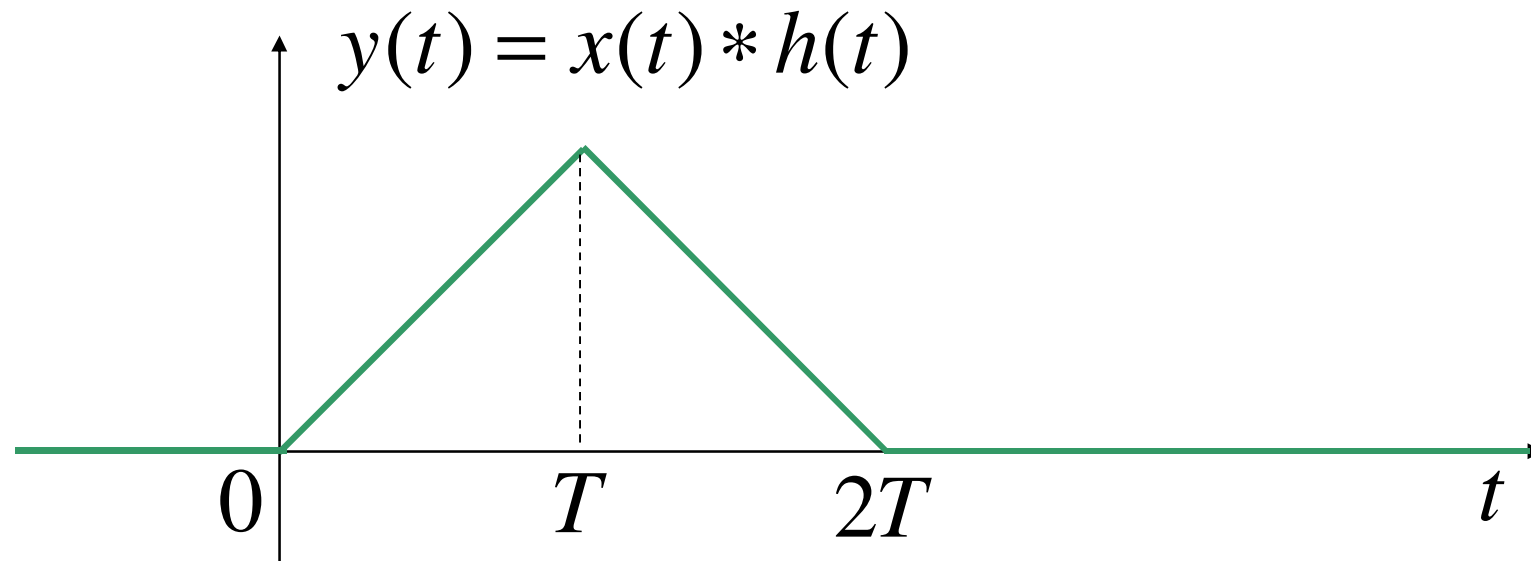
$$T \leq t - T \rightarrow 2T \leq t$$



$$y(t) = 0$$



Example of CT convolution



Applet 10



Properties of Convolution and CT LTI systems

1. A CT LTI system is completely characterized by its unit sample response:

Ex. 1. $h(t) = \delta(t - t_0)$

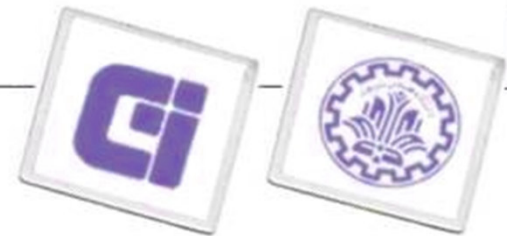
There are many systems with this response to $\delta(t)$.

There is only one LTI system with this response to $\delta(t)$:

$$y(t) = x(t - t_0)$$



$$x(t) * \delta(t - t_0) = x(t - t_0)$$

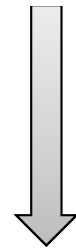


Properties of Convolution and CT LTI systems

Ex. 2. $y(t) = \int_{-\infty}^t x(\tau) d\tau$ - An Integrator

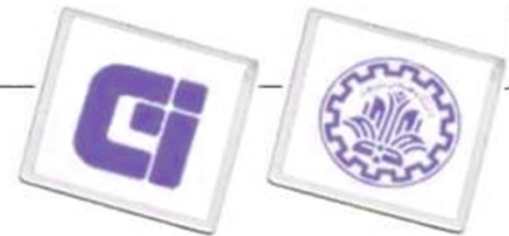
impulse response:

$$h(t) = \int_{-\infty}^t \delta(\tau) d\tau = u(t)$$



$$\begin{cases} x(t) = \delta(t) \\ y(t) = h(t) \end{cases}$$

$$y(t) = x(t) * h(t) = x(t) * u(t) = \int_{-\infty}^t x(\tau) d\tau$$



The Commutative Property

$$y(t) = x(t) * h(t) = h(t) * x(t)$$



Ex. Step response $s(t)$ of an LTI System:

$$s(t) = u(t) * h(t) = h(t) * u(t)$$

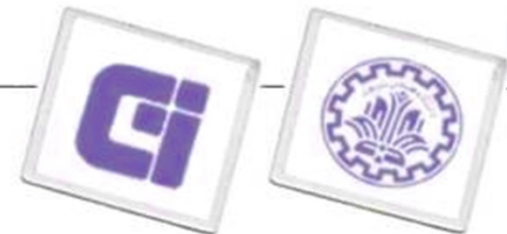
That is:

↑
Step Input

↑
"Input"

↑
Unit Sample response of
integrator

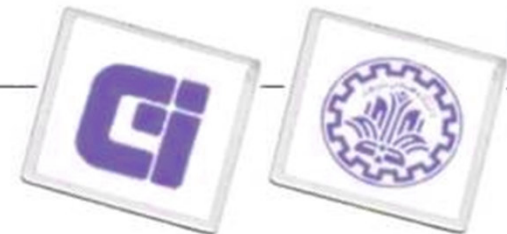
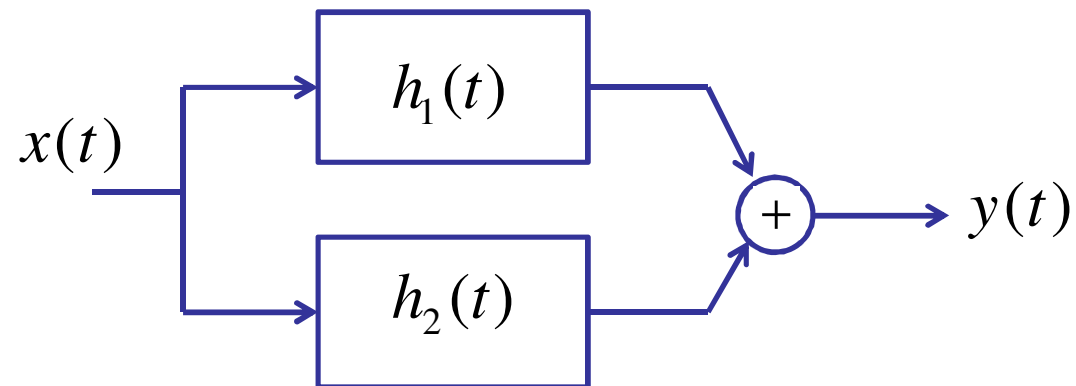
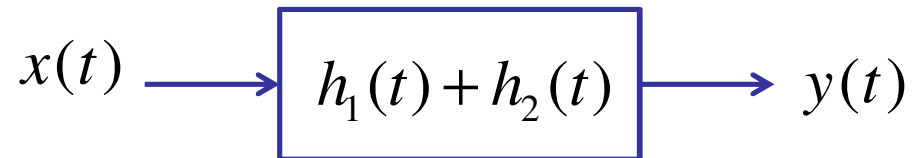
$$s(t) = \int_{-\infty}^t h(\tau) d\tau$$



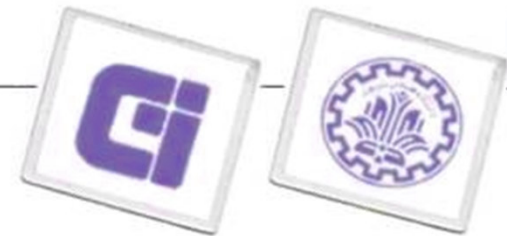
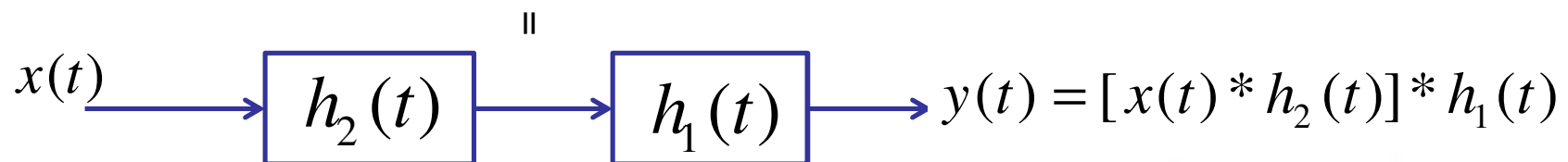
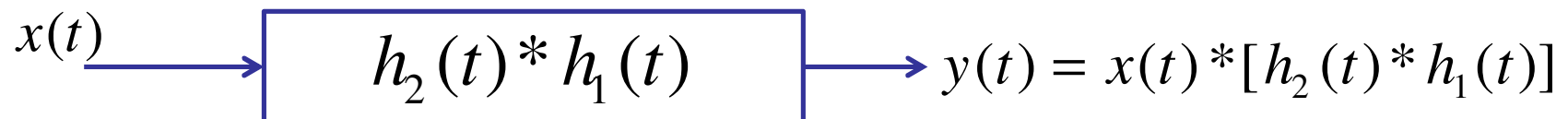
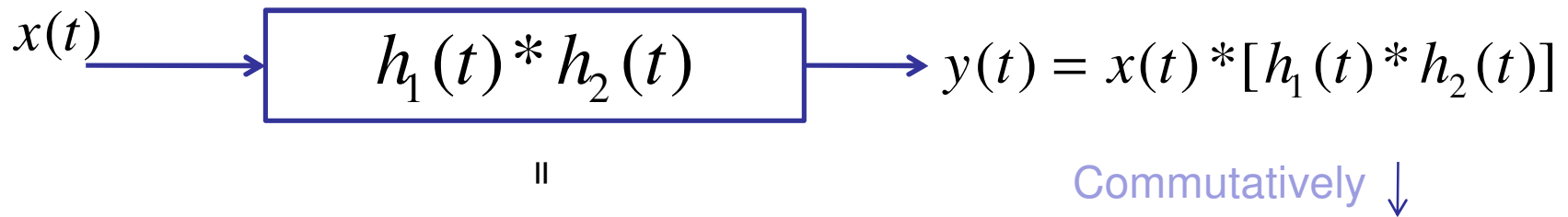
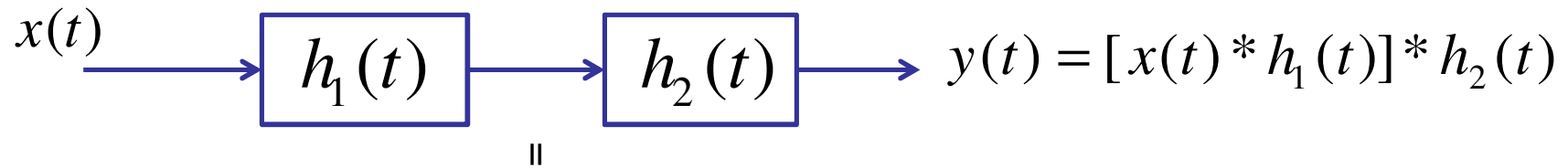
The Distributive Property

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

Interpretation:



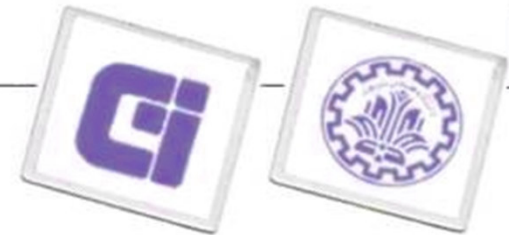
The Associative Property



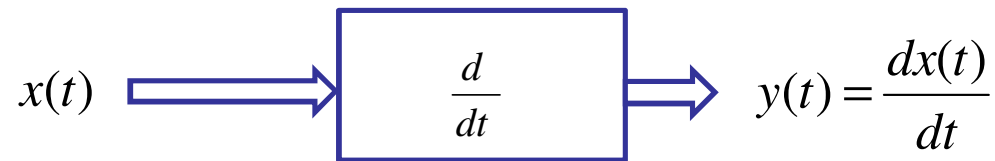
Properties of LTI Systems

1. **Causality** \iff $h(t)=0$ for all $t<0$

2. **Stability** \iff $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$



The Unit Doublet — Differentiator

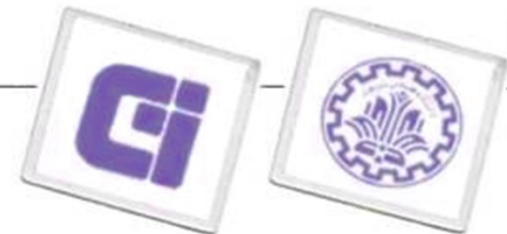


Impulse response = unit doublet

$$u_1(t) = \frac{d\delta(t)}{dt} \qquad u_0(t) = \delta(t)$$

The operational definition of the unit doublet:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$



Differentiators

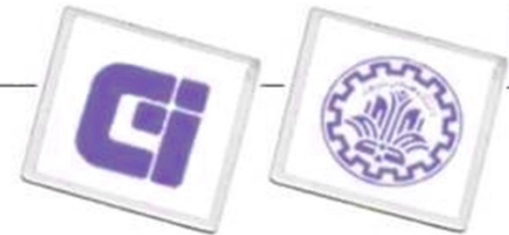
$$n > 0$$

$$u_n(t) = \underbrace{u_1(t) * \dots * u_1(t)}_{n \text{ times}}$$

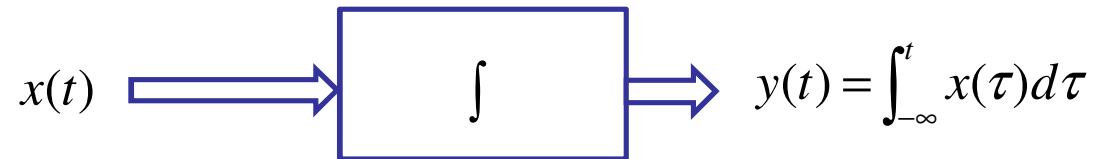
n is number of differentiations

Operational definitions

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \quad (n > 0)$$



Integrators



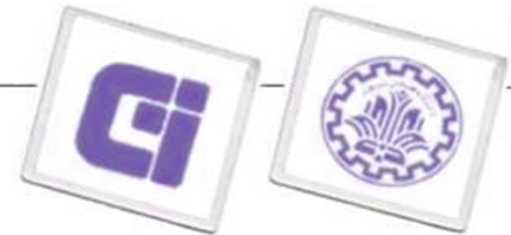
Impulse response: $u_{-1}(t) \equiv u(t)$

“-1 derivatives” = integral \Rightarrow I.R. = unit step

Operational definition: $x(t) * u_{-1}(t) = \int_{-\infty}^t x(\tau) d\tau$

Cascade of n integrators :

$$u_{-n}(t) = \underbrace{u_{-1}(t) * \dots * u_{-1}(t)}_{n \text{ times}}, \quad (n > 0)$$

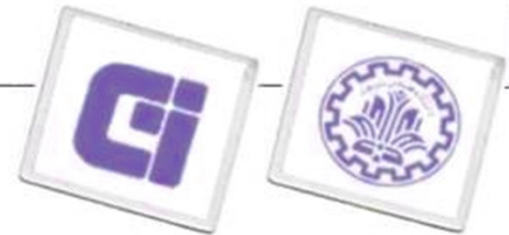


Integrators

$$\begin{aligned}u_{-2}(t) &= \int_{-\infty}^t u_{-1}(\tau) d\tau = \int_{-\infty}^t u(\tau) d\tau \\ &= t \cdot u(t)\end{aligned}$$

More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!} u(t)$$



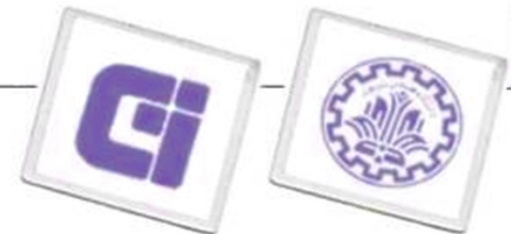
Notation

It may easily be seen that

$$u_n(t) * u_m(t) = u_{n+m}(t)$$

n and m may be positive or
negative

E.g. $u_1(t) * u_{-1}(t) = u_0(t)$



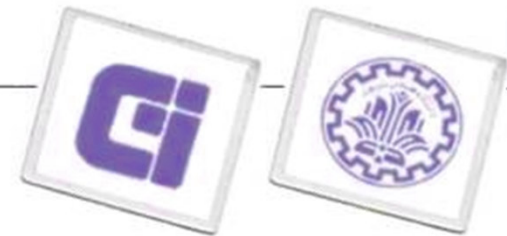
Linear-Constant-Coefficient Difference Equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$h[n]=?$

FT

$$\sum_{k=0}^N a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^M b_k e^{-jk\omega} X(e^{j\omega})$$



Linear-Constant-Coefficient Difference Equations

$$\rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^M b_k e^{-jk\omega}}{\sum_{k=0}^N a_k e^{-jk\omega}}$$

Zeros
Poles

Applet 18



Example of LCCDE

$$y[n] = \frac{1}{2} \{ x[n] - x[n-1] \}$$

$$h[n] = \frac{1}{2} \{ \delta[n] - \delta[n-1] \}$$



Edge detector



Edge enhancement using DT differentiator

