Digital Media Laboratory
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Signals & Systems

LTI Systems (CT Convolution & Difference Equations)

Adapted from: Lecture notes from MIT

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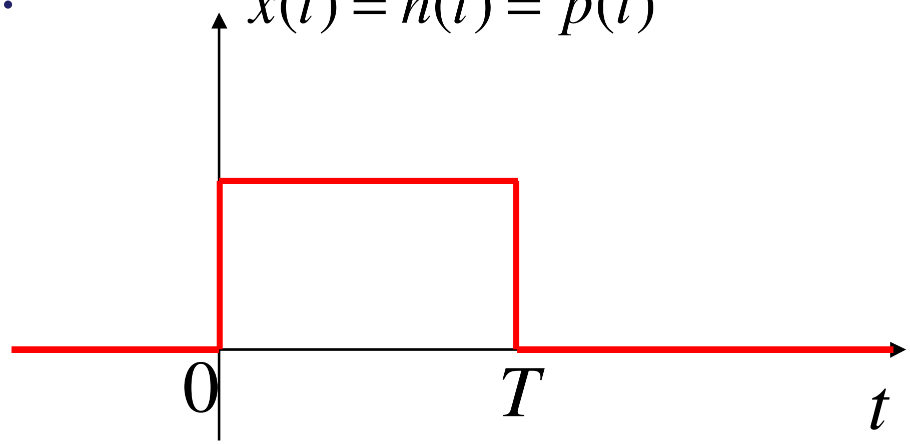
Example of CT convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Suppose that:

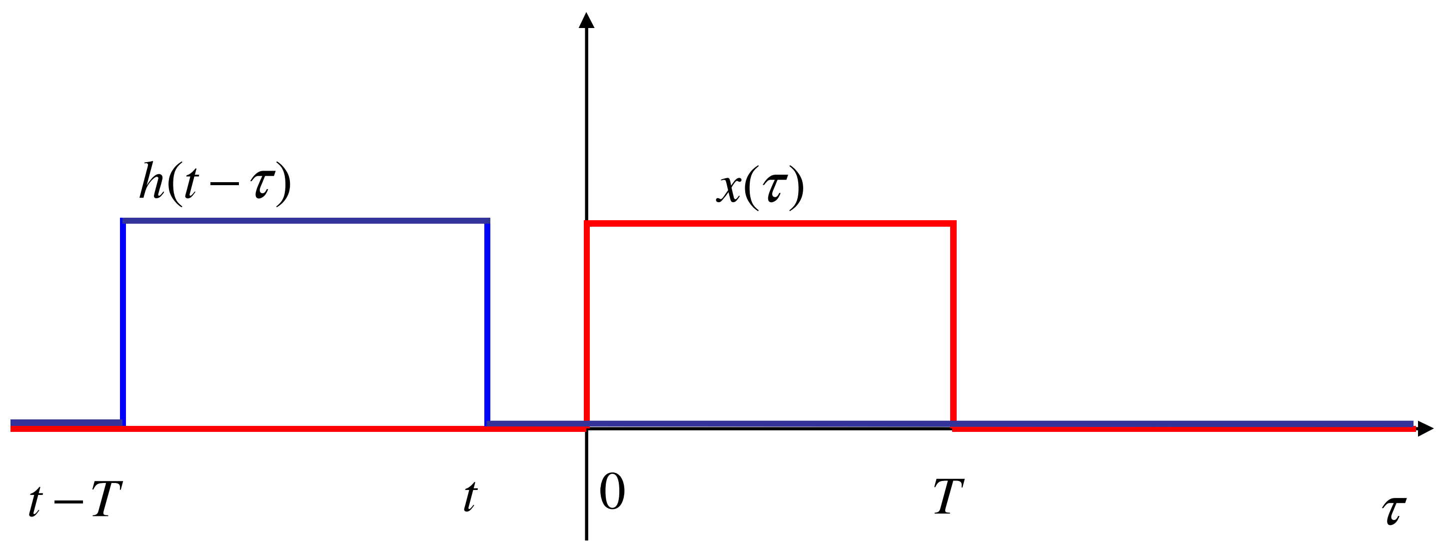
$$x(t) = h(t) = p(t)$$



We have to consider four cases.

Example of CT convolution

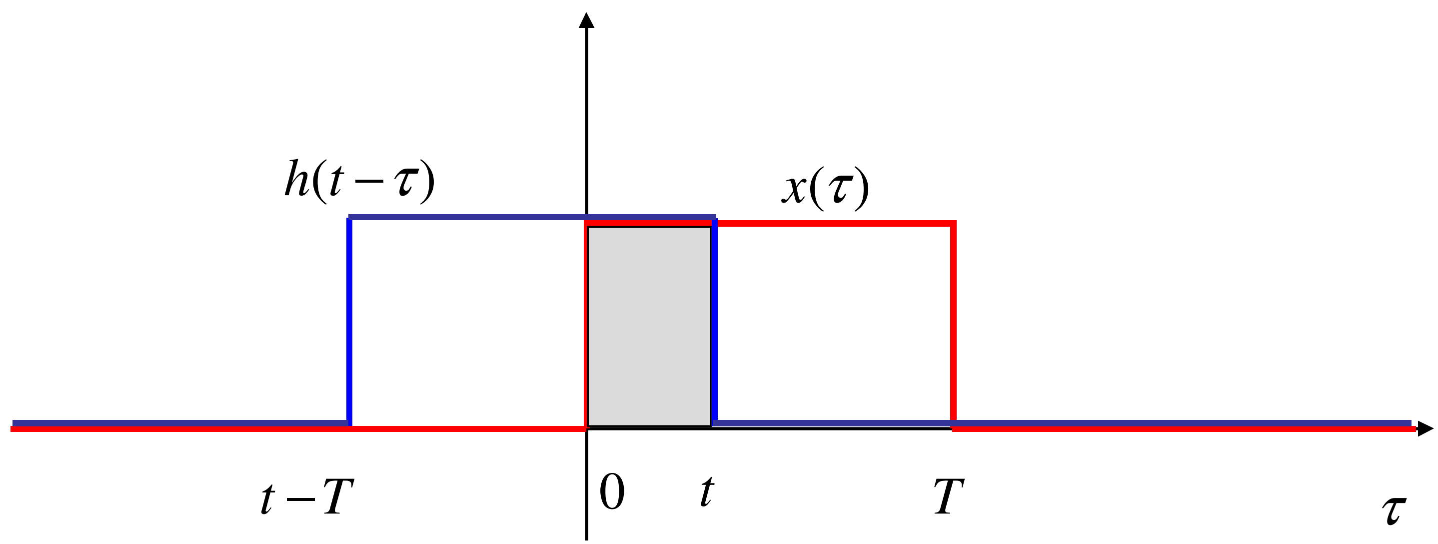
- **Case 1:** $t \leq 0$



$$y(t) = 0$$

Example of CT convolution

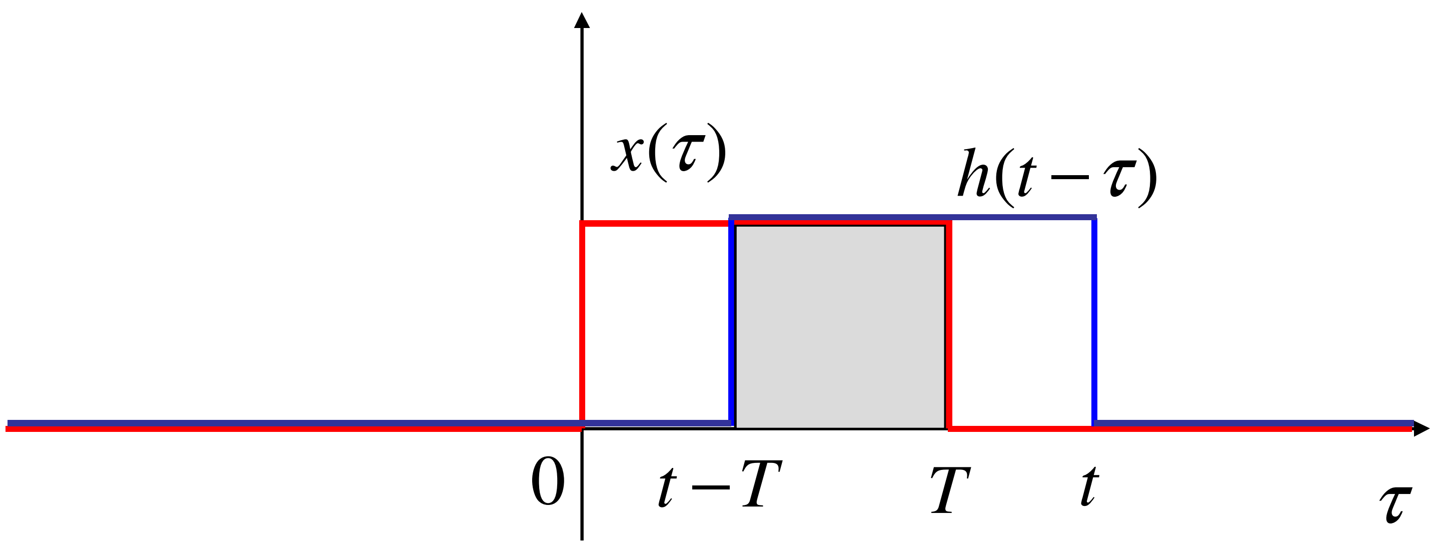
✧ **Case 2:** $0 \leq t \leq T$



$$y(t) = \int_0^t d\tau = t$$

Example of CT convolution

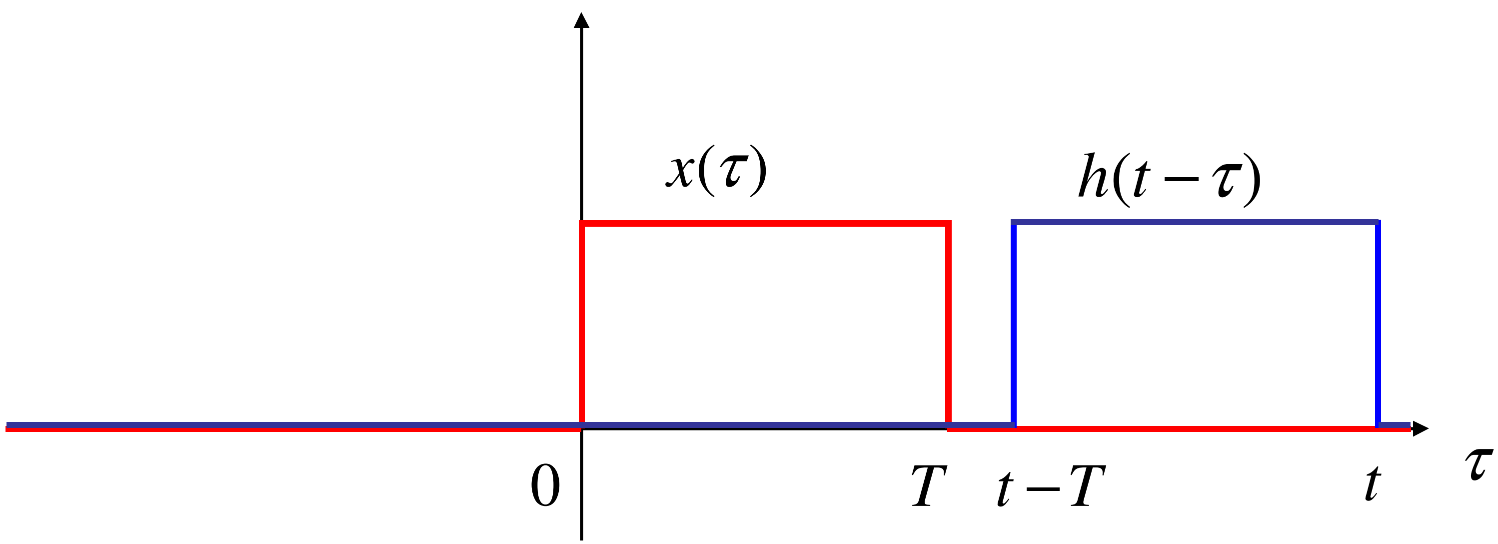
- **Case 3:** $0 \le t - T \le T \quad \rightarrow \quad T \le t \le 2T$



$$y(t) = \int_{t-T}^{T} d\tau = T - (t - T) = 2T - t$$

Example of CT convolution

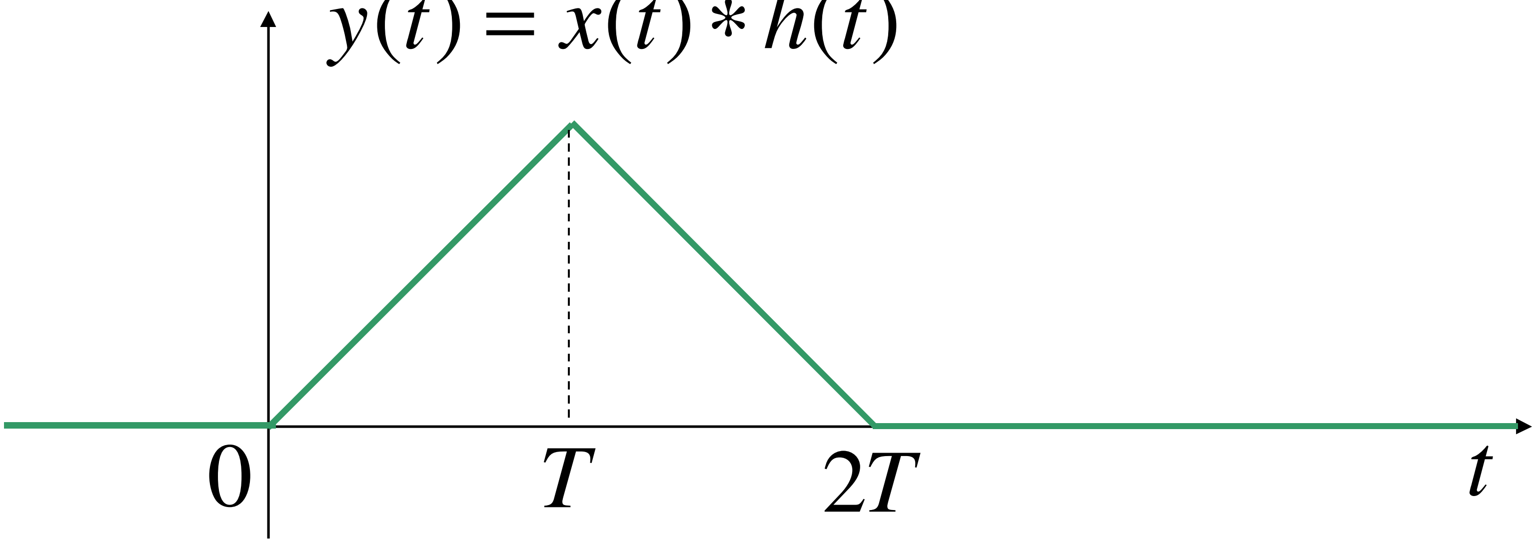
- **Case 4:** $$T \le t - T \quad \rightarrow \quad 2T \le t$$



$$y(t) = 0$$

Example of CT convolution

$$y(t) = x(t) * h(t)$$



Applet 10

Template designed by Jafar Muhammadi

Properties of Convolution and CT LTI systems

1. A CT LTI system is **completely characterized** by its unit sample response:

   Ex. 1. $h(t) = \delta(t - t_0)$

   There are **many systems** with this response to $\delta(t)$.

   There is **only one** LTI system with this response to $\delta(t)$:

$$y(t) = x(t - t_0)$$

$$\Downarrow$$

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

Properties of Convolution and CT LTI systems

Ex. 2. $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ **- An Integrator**

impulse response:

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$$

$$\Big\Downarrow \quad \begin{cases} x(t) = \delta(t) \\ y(t) = h(t) \end{cases}$$

$$y(t) = x(t) * h(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

The Commutative Property

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ $\qquad$ $h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$

Ex. Step response $s(t)$ **of an LTI System:**

$$s(t) = \underset{\substack{\uparrow \\ \text{Step Input}}}{u(t)} * h(t) = \underset{\substack{\uparrow \\ \text{"Input"}}}{h(t)} * \underset{\substack{\uparrow \\ \text{Unit Sample response of integrator}}}{u(t)}$$

That is:

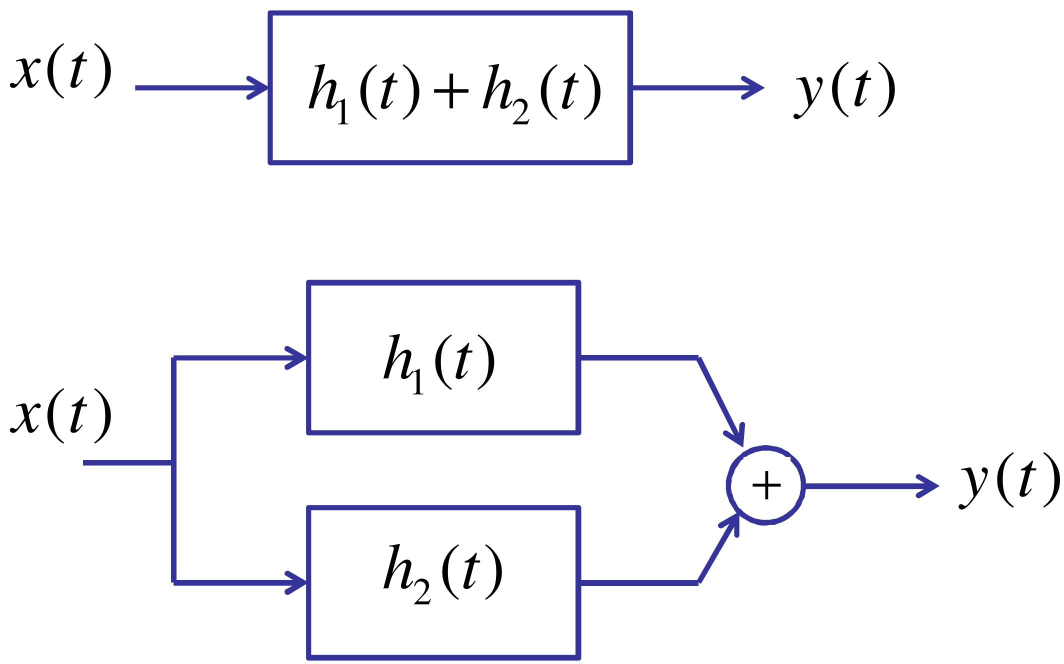
$$\Downarrow$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

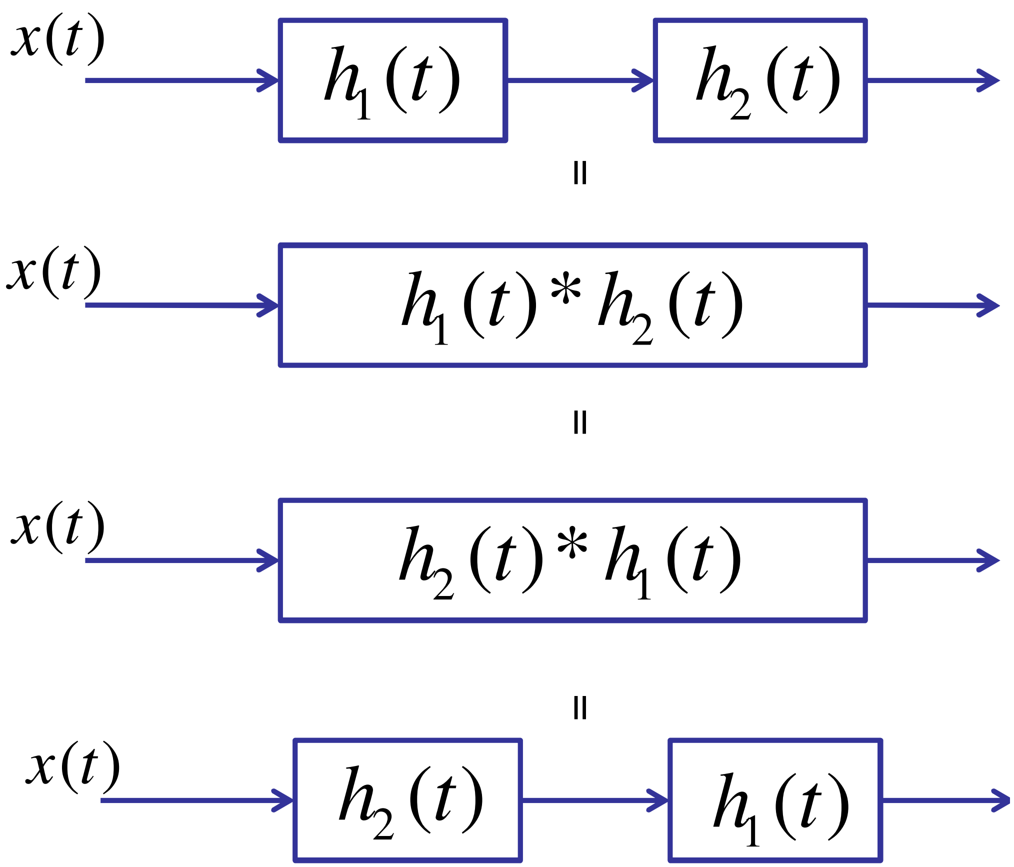
The Distributive Property

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

Interpretation:



The Associative Property



$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

$$y(t) = x(t) * [h_1(t) * h_2(t)]$$

Commutatively ↓

$$y(t) = x(t) * [h_2(t) * h_1(t)]$$

$$y(t) = [x(t) * h_2(t)] * h_1(t)$$

Properties of LTI Systems

1. **Causality** $\Longleftrightarrow$ h(t)=0 for all t<0

2. **Stability** $\Longleftrightarrow$ $\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$

The Unit Doublet —Differentiator

$$x(t) \Rightarrow \boxed{\frac{d}{dt}} \Rightarrow y(t) = \frac{dx(t)}{dt}$$

Impulse response = unit doublet

$$u_1(t) = \frac{d\delta(t)}{dt} \qquad\qquad u_0(t) = \delta(t)$$

The operational definition of the unit doublet:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

Differentiators

$$n > 0$$

$$u_n(t) = \underbrace{u_1(t) * \ldots * u_1(t)}_{\text{n times}}$$

n is number of differentiations

Operational definitions

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \quad (n > 0)$$

Integrators

$$x(t) \Rightarrow \boxed{\int} \Rightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Impulse response: $u_{-1}(t) \equiv u(t)$

"-1 derivatives" = integral ⇒I.R.= unit step

Operational definition: $$x(t) * u_{-1}(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Cascade of n integrators :

$$u_{-n}(t) = \underbrace{u_{-1}(t) * \ldots * u_{-1}(t)}_{\text{n times}}, \quad (n > 0)$$

Integrators

$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)d\tau = \int_{-\infty}^{t} u(\tau)d\tau$$

$$= t.u(t)$$

More generally, for n>0

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!}u(t)$$

Template designed by Jafar Muhammadi

Notation

It may easily be seen that

$$u_n(t) * u_m(t) = u_{n+m}(t)$$

n and m may be positive or negative

E.g. $$u_1(t) * u_{-1}(t) = u_0(t)$$

Linear-Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

h[n]=?

FT

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

Linear-Constant-Coefficient Difference Equations

$$\Rightarrow \quad H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Zeros
Poles

Applet 18

Example of LCCDE

$$y[n] = \frac{1}{2}\{x[n] - x[n-1]\}$$

$$h[n] = \frac{1}{2}\{\delta[n] - \delta[n-1]\}$$

Edge detector

Edge enhancement using DT differentiator