Digital Media Laboratory
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Signals & Systems

Fourier Series (Part I)

Adapted from: Lecture notes from MIT

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Transformation

✧ **General form:**

$$x(t) = \sum_{i=-\infty}^{\infty} a_i \varphi_i(t)$$

Coefficient (a_i)

Basis Function ($\varphi_i(t)$)

Fourier

Laplace

Karhunen-Loeve

Z-Transform

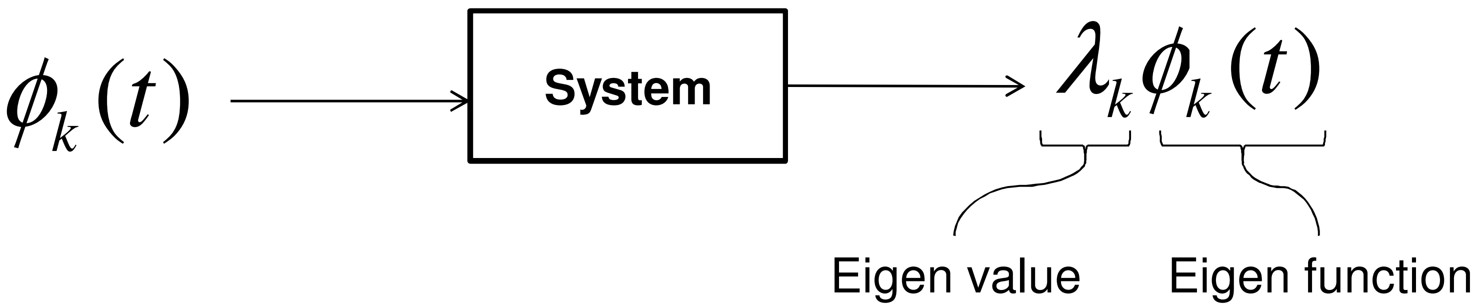
Wavelet

Desirable Characteristics of a Set of "Basic" Signals

- a) We can represent large and useful classes of signals using these building blocks.
- b) The response of LTI systems to these basic signals is particularly simple , useful and insightful.

- Previous focus: Unit samples and impulses
- Focus now: Eigen functions of all LTI systems

The eigenfunctions $\varphi_k(t)$ and their properties

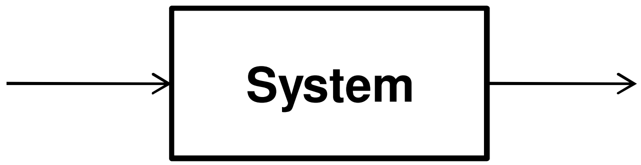


Eigenfunction in → Same function out with a "gain"

The eigenfunctions $\varphi_k(t)$ and their properties

From the superposition property of LTI system

$$x(t) = \sum_k a_k \phi_k(t)$$

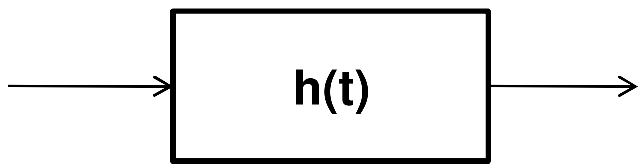


$$y(t) = \sum_k \lambda_k a_k \phi_k(t)$$

Now the task of finding response of LTI systems is to determine λ_k

Complex Exponentials as the Eigen functions of any LTI Systems

$$x(t) = e^{st}$$



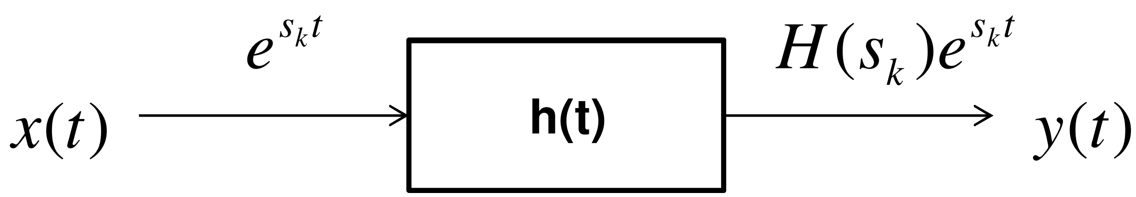
$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= \left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st}$$

$$= \underbrace{H(s)}_{\text{Eigen value}} \underbrace{e^{st}}_{\text{Eigen function}}$$

Eigen value   Eigen function

Complex Exponentials as the Eigen functions of any LTI Systems



$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

$$x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k H(s_k) a_k e^{s_k t}$$

Complex Exponentials as the Eigen functions of any LTI Systems

$x[n] = z^n$ → **h[n]** →

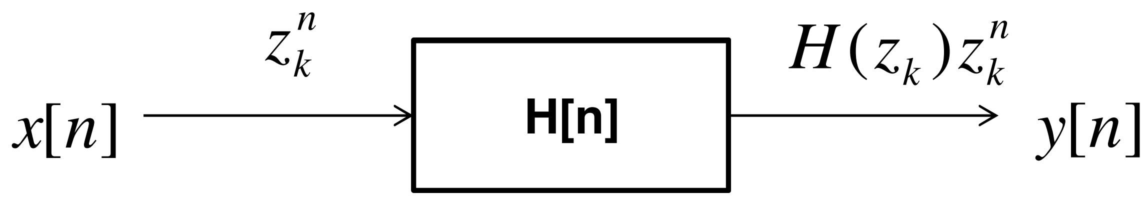
$$y[n] = \sum_{m=-\infty}^{\infty} h[m] z^{n-m}$$

$$= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right] z^n$$

$$= \underbrace{H(z)}_{\text{Eigen value}} \underbrace{z^n}_{\text{Eigen function}}$$

Eigen value Eigen function

Complex Exponentials as the Eigen functions of any LTI Systems



$$H(z) = \sum_{-\infty}^{\infty} h[n] z^{-n}$$

$$x[n] = \sum_k a_k z_k^n \rightarrow y[n] = \sum_k H(z_k) a_k z_k^n$$

What kinds of signals can we represent as "**sums**" of complex exponentials?

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For Now: **Focus on restricted sets of complex exponentials**

CT: $s = j\omega$

signals of the form $e^{j\omega t}$

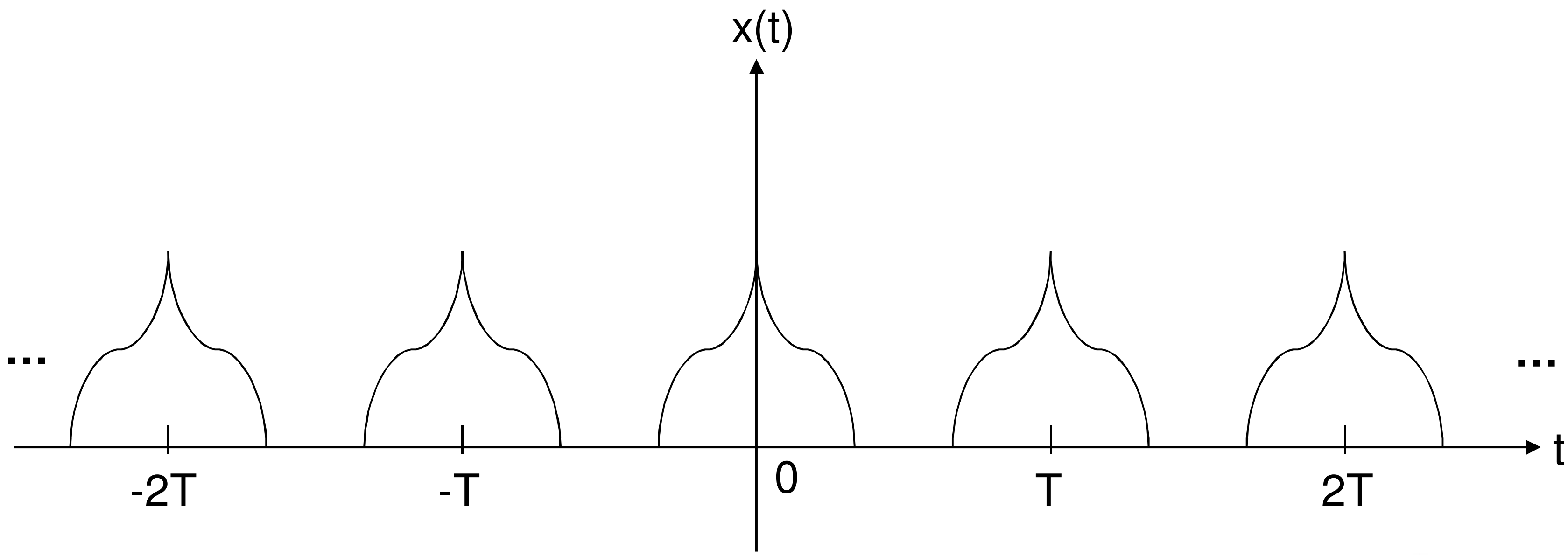
DT: $Z = e^{j\omega}$

signals of the form $e^{j\omega n}$

Fourier Series Representation of CT Periodic Signals

$$x(t) = x(t+T) \text{ for all } t$$

- Smallest such T is the fundamental period
- $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency



Fourier Series Representation of CT Periodic Signals

$x(t) = e^{j\omega t}$ Periodic with period T

$$\omega = k\omega_0 \Rightarrow x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

- Periodic with period T
- $\{a_k\}$ are the Fourier (series) Coefficients
- k=0: DC
- |k|=1: First Harmonic
- |k|=2: Second Harmonic

Question #1: How do we find the Fourier coefficients?

$$\text{Example 1}: x(t) = \cos 4\pi t + 2\sin 8\pi t$$

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$$\text{Example 1}: x(t) = \cos 4\pi t + 2\sin 8\pi t$$

$$x(t) \overset{\text{Euler's Realtion}}{=} \frac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \frac{2}{2j}[e^{j8\pi t} - e^{-j8\pi t}]$$

$$\omega_0 = 4\pi \Rightarrow T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

- $a_0 = 0$: no DC component
- $a_1 = \frac{1}{2}$
- $a_{-1} = \frac{1}{2}$
- $a_2 = \frac{1}{j}$
- $a_{-2} = -\frac{1}{j}$

Fourier Series Representation of CT Periodic Signals

For real periodic signals, there are two other commonly used forms for CT Fourier series:

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t]$$

or

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\gamma_k \cos(k\omega_0 t + \Theta_k)]$$

- Because of the Eigen function property of $e^{j\omega t}$, we will usually use the complex exponential form in this course
- A consequence of this is that we need to include terms for both positive and negative frequencies: $e^{jk\omega_0 t}, e^{-jk\omega_0 t}$

Question #1: How do we find the Fourier coefficients?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

1. **Multiply by $e^{-jn\omega_0 t}$**
2. **Integrate over one period**

$$\int_T x(t) e^{-jn\omega_0 t} dt = \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-jn\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_T e^{j(k-n)\omega_0 t} dt \right)$$

Here $\int_T$ denotes integral over interval of length T (one period).

$$\int_T e^{-j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} = T\delta[k-n]$$

Question #1: How do we find the Fourier coefficients?

$$\int_T x(t)e^{-jn\omega_0 t}dt = \sum_{-\infty}^{\infty} a_k \left(\int_T e^{-j(k-n)\omega_0 t}dt \right) = \sum_{-\infty}^{\infty} a_k T\,\delta[k-n]$$

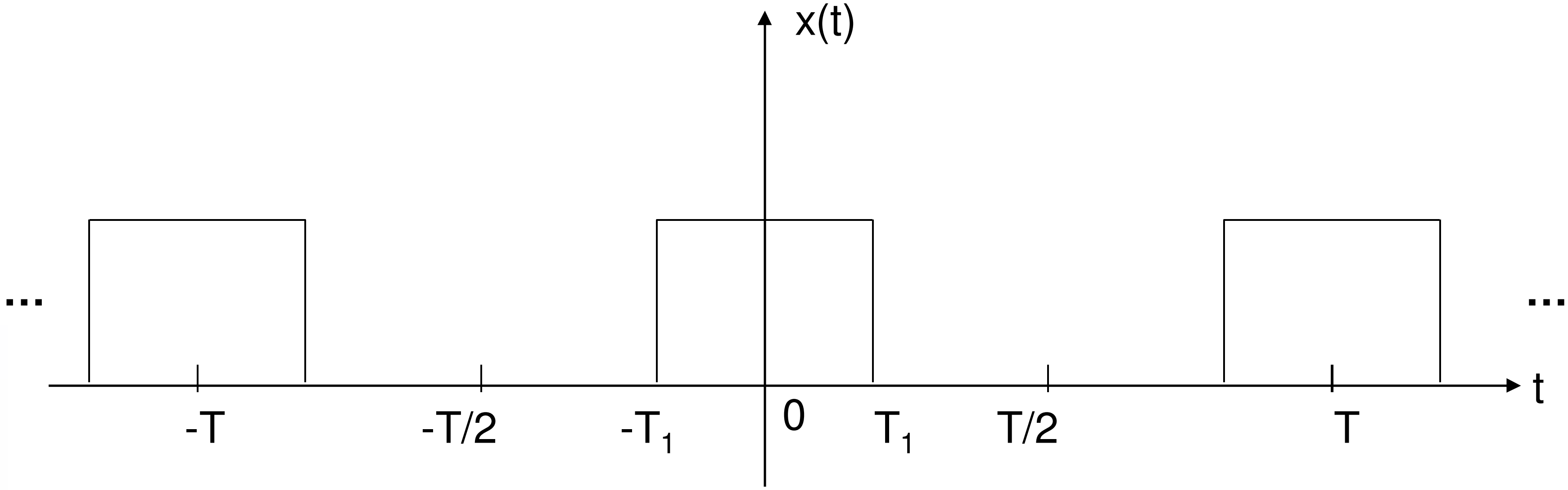
$$\int_T x(t)e^{-jn\omega_0 t}dt = a_n T$$

CT Fourier Series Pair ($\omega_0 = \frac{2\pi}{T}$)

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ Synthesis Equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$ Analysis Equation

Example #2: Periodic Square Wave



For k = 0

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{2T_1}{T}$$

DC component is just the average

Example #2: Periodic Square Wave

For k ≠ 0

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt$$

$$= -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{\sin k\omega_0 T_1}{k\pi} \quad (\omega_0 = \frac{2\pi}{T})$$

Convergence of CT Fourier Series

- How can the Fourier series for the square wave possibly make sense?
- The key is: What do we mean by $e(t) = x(t) - \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$?
- One useful notion for engineers
  - There is no energy in the difference

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \int_T |e(t)|^2 \, dt = 0$$

- Just need x(t) to have finite energy per period.

$$\int_T |x(t)|^2 \, dt < \infty$$

Dirichlet Conditions

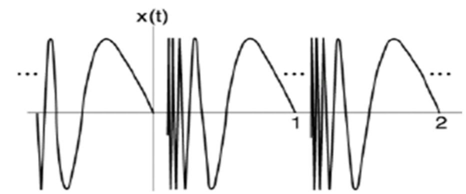
- The Dirichlet conditions are sufficient conditions for a real-valued, periodic function $f(x)$ to be equal to the sum of its Fourier series at each point where f is continuous.
- The behaviour of the Fourier series at points of discontinuity is determined as well, by these conditions.
- These conditions are named after Johann Peter Gustav Lejeune Dirichlet.

Dirichlet Conditions

- **Condition 1:** **x(t) is absolutely integrable over one period, i. e.**

$$\int_T |x(t)|\,dt < \infty$$

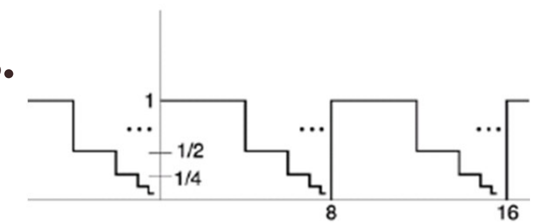
- **Condition 2:** **In a finite time interval, x(t) has a finite number of maxima and minima.**



Ex. **An example that violates Condition 2.**

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \qquad 0 < t \le 1$$

- **Condition 3:** **In a finite time interval, x(t) has only a finite number of discontinuities.**



Ex. **An example that violates Condition 3.**

Dirichlet Conditions

- **Dirchlet Conditions** are met for the signals we will encounter in the real world. Then:
  - The Fourier series = x(t) at points where x(t) is continuous.
  - The Fourier series = "midpoint" at points of discontinuity.
- Still, convergence has some interesting characteristics:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

  - As $N \rightarrow \infty$, $x_N(t)$ exhibits *Gibbs'* phenomenon at points of discontinuity.

Applet 15

Gibbs Phenomenon

- Fourier sums overshoot at a jump discontinuity
- This overshoot does not die out as the frequency increases.

$$S_N f(x) = \sin x + \frac{1}{3}\sin(3x) + \ldots + \frac{1}{N-1}\sin((N-1)x)$$