Digital Media Laboratory
Advanced Information & Communication Technology Center
Sharif University of Technology

Signals & Systems

Fourier Series (Part II)

Adapted from: Lecture notes from MIT

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CT Fourier Series Pairs

Review

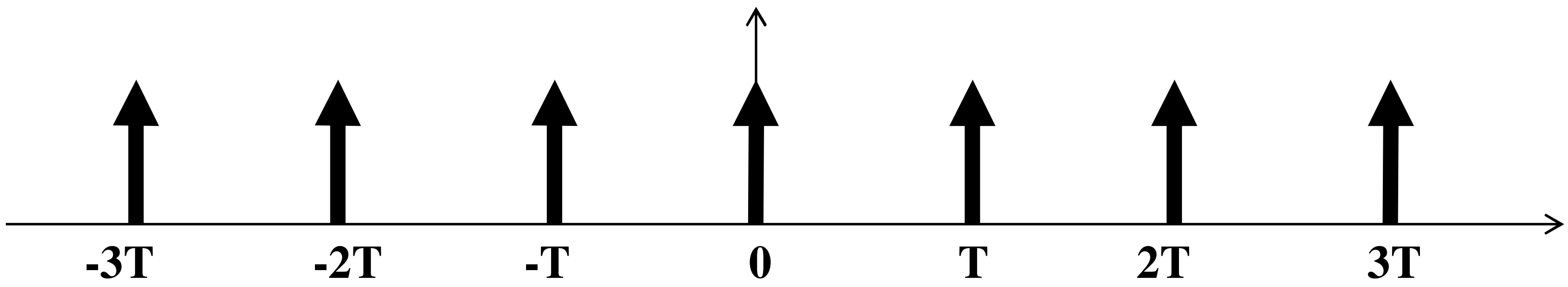
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \overset{\omega_0 = \frac{2\pi}{T}}{=} \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

$$x(t) \xrightarrow{FS} a_k$$

Example #1: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



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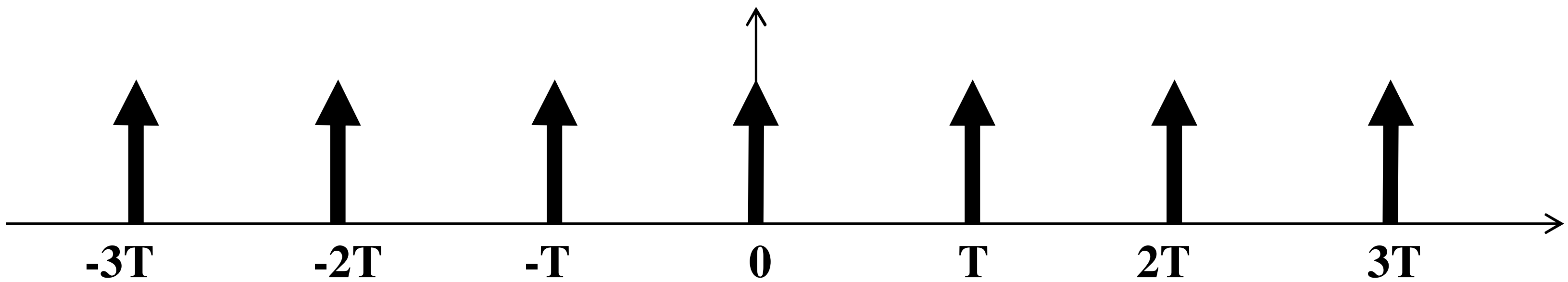
Example #1: Periodic Impulse Train

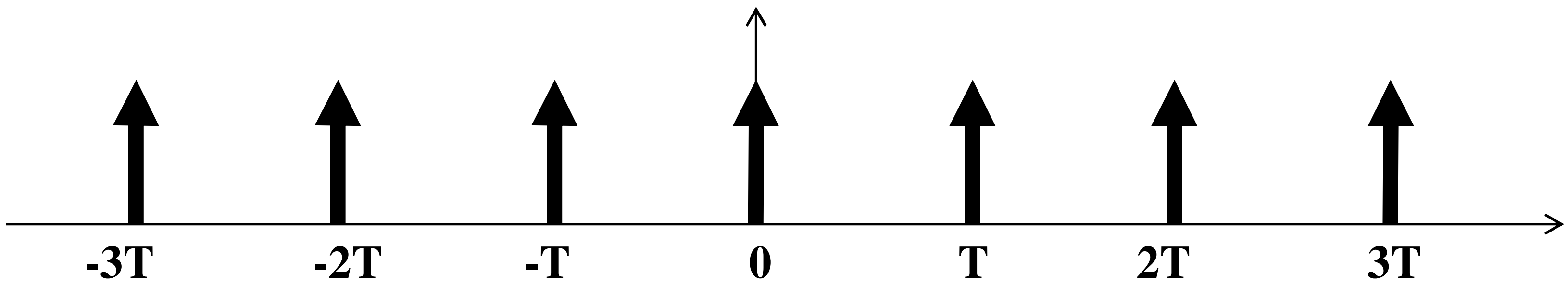
$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T} \text{ for all k!}$$

Example #1: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

All components have: 1)The same amplitude 2)The same phase

(A few of the) Properties of CT Fourier Series

- **Linearity:**

$$\left.\begin{array}{l} x(t) \leftrightarrow a_k \\ y(t) \leftrightarrow b_k \end{array}\right\} \Rightarrow \alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$$

- **Conjugate Symmetry**

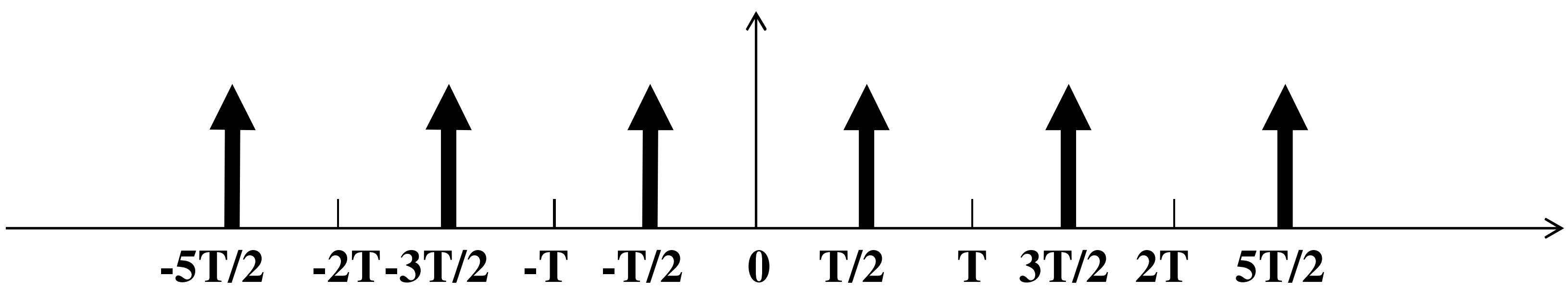
$$\left.\begin{array}{r} x(t) \text{ is real} \Rightarrow a_{-k} = a_k^* \\ a_k = \operatorname{Re}\{a_k\} + j\operatorname{Im}\{a_k\} = |a_k| e^{j\angle a_k} \end{array}\right\} \Rightarrow \begin{array}{c} \operatorname{Re}\{a_k\} \text{ is even, } \operatorname{Im}\{a_k\} \text{ is odd} \\ |a_k| \text{ is even, } \angle a_k \end{array}$$

- **Time Shift**

$$x(t) \leftrightarrow a_k \Rightarrow x(t - t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0} = a_k e^{-jk\frac{2\pi}{T}t_0}$$

Example #2: Shift by half period

$$y(t) = x(t-\frac{T}{2}) = \sum_{n=-\infty}^{\infty} \delta(t-nT-\frac{T}{2})$$



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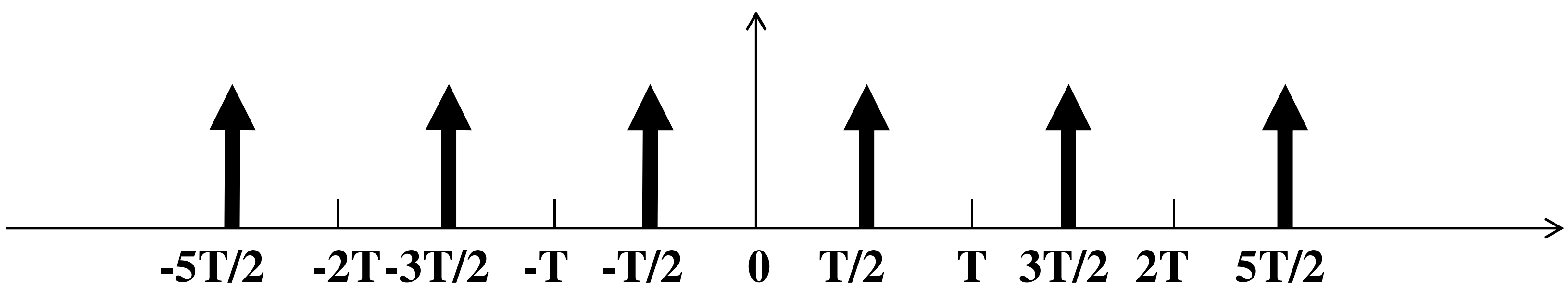
Example #2: Shift by half period

$$y(t) = x(t - \frac{T}{2}) = \sum_{n=-\infty}^{\infty} \delta(t - nT - \frac{T}{2})$$



Time Shift Propery

$$x(t) \leftrightarrow a_k \overset{\text{Time Shift Propery}}{\Rightarrow} y(t) \leftrightarrow a_k e^{-jk\omega_0 \frac{T}{2}} = a_k e^{-jk\frac{2\pi}{T}\frac{T}{2}} = a_k e^{-jk\pi}$$

$$= a_k (-1)^k \overset{a_k = \frac{1}{T}}{=} \frac{(-1)^k}{T}$$

Parseval's Relation

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2\, dt}_{\text{Average signal power}} = \sum_{k=-\infty}^{\infty} \underbrace{|a_k|^2}_{\text{Power in the } k_{th} \text{ harmonic}}$$

Energy is the same whether measured in the time-domain or the frequency-domain

Multiplication Property

$$\left.\begin{array}{c} x(t) \leftrightarrow a_k \\ y(t) \leftrightarrow b_k \\ \text{both x(t) and y(t) are periodic} \\ \text{with them same period T} \end{array}\right\} \Rightarrow x(t).y(t) \leftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$$

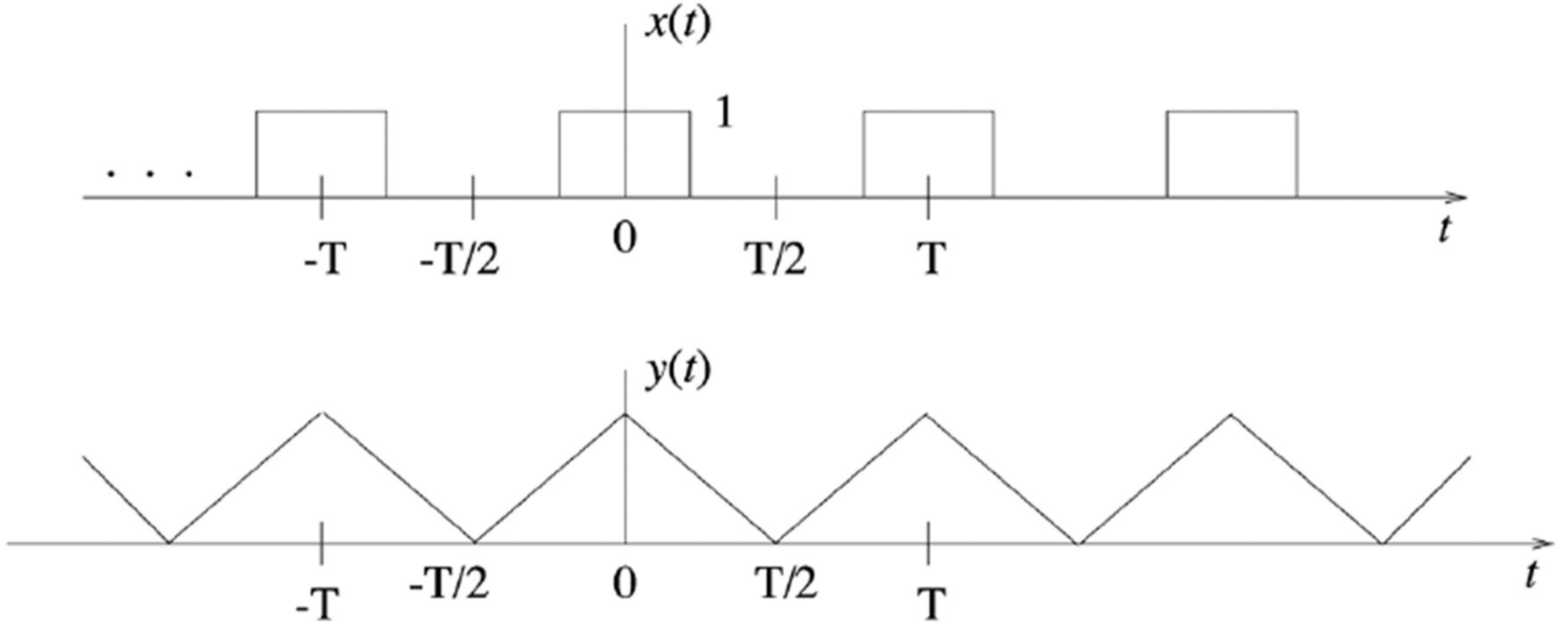
Proof:

$$x(t).y(t) = \sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t} . \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t} = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_l b_m e^{j(l+m)\omega_0 t}$$

$$\overset{l+m=k}{\Rightarrow} \sum_{k=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} a_l b_{k-l} \right] e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Rightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Periodic Convolution

x(t), y(t) periodic with period T



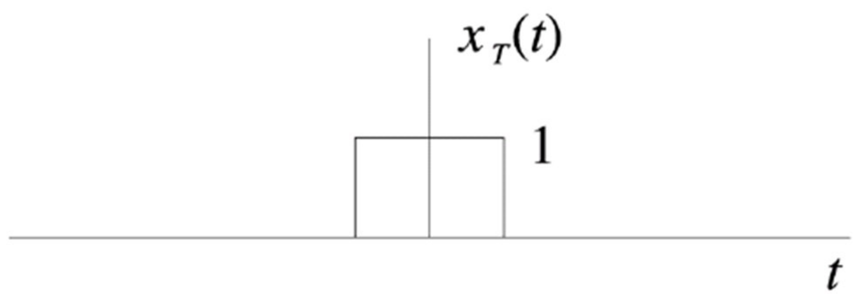
$$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau \quad \xrightarrow{\text{x(t) and y(t) are positive}} \quad x(t)*y(t)=\infty$$

Periodic Convolution

Periodic Convolution: Integrate over anyone period (e.g. –T/2 to T/2)

$$x(t) * y(t) = z(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)\, y(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x_T(\tau)\, y(t-\tau)\, d\tau$$

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} < t < \frac{T}{2} \\ 0 & otherwise \end{cases}$$



Periodic Convolution Facts

- $z(t)$ is periodic with period T (why?)
  From previous lectures: $x(t) = x(t + T) \rightarrow y(t) = y(t + T)$ for LTI systems.
  In the convolution, treat $y(t)$ as the input and $x_T(t)$ as $h(t)$
- Doesn't matter what period over which we choose to integrate:

$$z(t) = \int_T x(\tau)\, y(t-\tau)\, d\tau = x(t) \otimes y(t)$$

- Periodic Convolution in Time ↔ Multiplication in Frequency

$$x(t) \leftrightarrow a_k,\ y(t) \leftrightarrow b_k,\ x(t) \otimes y(t) = z(t) \leftrightarrow c_k$$

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T \left(\int_T x(\tau) y(t-\tau) d\tau \right) e^{-jk\omega_0 t} dt$$

$$= \int_T \left(\frac{1}{T}\int_T y(t-\tau) e^{-jk\omega_0 (t-\tau)} dt \right) x(\tau) e^{-jk\omega_0 \tau} d\tau = \int_T b_k x(\tau) e^{-jk\omega_0 \tau} d\tau = T a_k b_k$$

Fourier Series Representation of DT Periodic Signals

- x[n] is periodic with fundamental period N and fundamental frequency ω_0

$$x[n+N]=x[n] \text{ and } \omega_0=\frac{2\pi}{N}$$

- Only $e^{j\omega n}$ which are periodic with period N will apear in the *FS*

$$\omega N=k2\pi \Leftrightarrow \omega=k\omega_0, \qquad k=0,\pm1,\pm2,\ldots$$

- There are only N distinct signals of this form

$$e^{j(k+N)\omega_0 n}=e^{jk\omega_0 n}e^{jN\omega_0 n}\overset{N\omega_0 n=2\pi n}{=}e^{jk\omega_0 n}$$

- So we *could* just use $e^{j0\omega_0 n}, e^{j1\omega_0 n}, e^{j2\omega_0 n},\ldots, e^{j(N-1)\omega_0 n}$
- However, it is often useful to allow the choice of N consecutive values of k to be *arbitrary*.

DT Fourier Series Representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$\sum_{k=<N>}$ Sum over any N consecutive values of k

$\{a_k\}$ Fourier series coefficients

Questions:

1. **What DT periodic signals have such a representation?**
2. **How do we find a_k?**

DT Fourier Series Representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$\sum_{k=<N>}$ Sum over any N consecutive values of k

$\{a_k\}$ Fourier series coefficients

Questions #1: What DT periodic signals have such a representation?

Questions #2: How do we find a_k?

Answer to Question #1

Any DT periodic signal has a Fourier Series representation

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$$

$$\Downarrow$$

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jkw_0}$$

$$x[2] = \sum_{k=<N>} a_k e^{j2kw_0}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{j(N-1)kw_0}$$

N equations for N unknowns: $a_0, a_1, a_2, \ldots, a_N$

A More Direct Way to Solve for a_k

Finite geometric series

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & ,\alpha=1 \\ \dfrac{1-\alpha^N}{1-\alpha} & ,\alpha\neq 1 \end{cases}$$

$$\Downarrow \alpha = e^{jk\omega_0}$$

$$\sum_{n=<N>} e^{jk\omega_0 n} = \sum_{n=0}^{N-1} (e^{jk\omega_0})^n = \sum_{n=0}^{N-1} (e^{jk2\pi/N})^n$$

$$= \begin{cases} N & ,k=0,\pm N,\pm 2N,... \\ \dfrac{1-e^{jk(2\pi/N)N}}{1-e^{jk\omega_0}}=0 & ,otherwise \end{cases}$$

Template designed by Jafar Muhammadi

A More Direct Way to Solve for a_k

So, from $x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$ multiply both sides by $e^{-jm\omega_0 n}$ and then $\sum_{n=<N>}$

$$\sum_{n=<N>} x[n] e^{-jm\omega_0 n} = \sum_{n=<N>} \left(\sum_{k=<N>} a_k e^{jk\omega_0 n} \right) e^{-jm\omega_0 n}$$

$$= \sum_{k=<N>} a_k \underbrace{\left(\sum_{n=<N>} e^{j(k-m)\omega_0 n} \right)}_{N\delta[k-m]} = N a_m$$

orthogonality

DT Fourier Series Pair ($\omega_0 = \frac{2\pi}{T}$)

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} \qquad \text{Synthesis Equation}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} \qquad \text{Analysis Equation}$$

Note: It is convenient to think of a_k as being defined for *all* integers k. So:

1. **$a_{k+N} = a_k$: Special property of DT Fourier Coefficients.**
2. **We only use N consecutive values of a_k in the synthesis equation.** **(Since $x[n]$ is periodic, it is specified by N numbers, either in the time or frequency domain)**

Template designed by Jafar Muhammadi

Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n / 8) + \cos(\pi n / 4 + \pi / 4)$$

(periodic with period N = 16 → $\omega_0 = \pi/8$)

$$x[n] = \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}] + \frac{1}{2}[e^{j\pi/4} e^{j2\omega_0 n} + e^{-j\pi/4} e^{-j2\omega_0 n}]$$

$$\Downarrow$$

$a_0 = 0$

$a_1 = 1/2$

$a_{-1} = 1/2$

$a_2 = e^{j\pi/4}/2$

$a_{-2} = e^{-j\pi/4}/2$

$a_3 = 0$

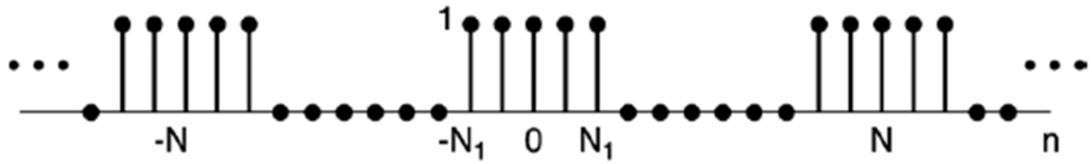
$a_{-3} = 0$

$\vdots$

$$a_{15} = a_{-1+16} = a_{-1} = \frac{1}{2}$$

$$a_{66} = a_{2+4\times16} = a_2 = e^{j\pi/4}/2$$

Example #2: DT Square Wave



$$a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] = \frac{2N_1+1}{N} = a_N = a_{-N} = a_{6N} = \cdots$$

For k ≠ multiple of N: (Using n = m – N_1)

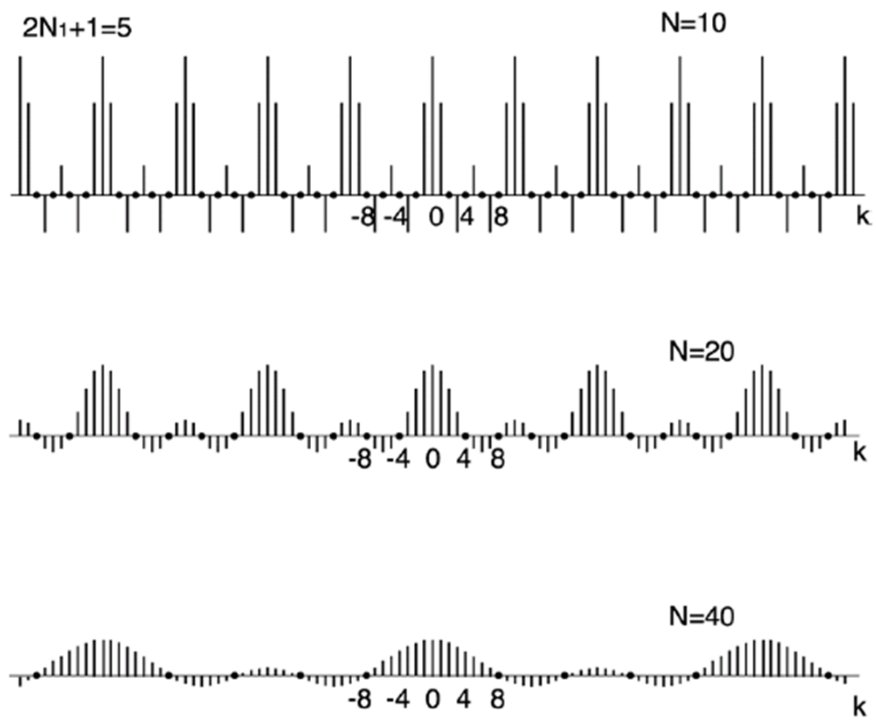
$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\omega_0(m-N_1)}$$

$$= \frac{1}{N} e^{jk\omega_0 N_1}\sum_{m=0}^{2N_1}(e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1}\frac{1-e^{-jk\omega_0(2N_1+1)}}{1-e^{jk\omega_0}}$$

$$= \frac{1}{N}\frac{\sin[k(N_1+1/2)\omega_0}{\sin(k\omega_0/2)} = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}$$

Example #2: DT Square Wave

$$a_k = \frac{1}{N}\frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}$$



Convergence Issues for DT Fourier Series:

Not an issue, since all series are finite sums.

Properties of DT Fourier Series: **Lots, just as with CT Fourier Series**

Example:

$$x[n] \leftrightarrow a_k$$

$$e^{jM\omega_0 n} x[n] \leftrightarrow b_k = ?$$

$$x[n]e^{jM\omega_0 n} = \sum_{r=<N>} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$

$$\underline{\underline{k = r + m}} \sum_{r=<N>} a_{k-M} e^{jk\omega_0 n}$$

$$\Downarrow$$

$$b_k = a_{k-M}$$

Frequency shift

$$jk\omega_0 \rightarrow j(k-M)\omega_0$$