In The Name of Allah

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Solution of Homework 5

1. Reference Book Solution Manual, Page 45, Problem 21

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2. Reference Book Solution Manual, Page 46, Problem 22

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3. (a) $x[n]$ is an aperiodic signal with extent $[0, N-1]$. The periodic signal

$$\tilde{y}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$$

is periodic with period N. To get the Fourier series coefficients for $\tilde{y}[n]$, we sum over one period of $\tilde{y}[n]$ to get

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)} n$$

(b) The Fourier transform of $x[n]$ is

$$X(\Omega) = \sum_{n=-\infty}^{infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega} n$$

since $x[n] = 0$for $n < 0,\ n > N-1$.

We can now easily see the relation between a_k and $XOmega$ since

$$\frac{1}{N} X(\Omega)|_{\Omega=(2\pi k)/N} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n}$$

Therefore,

$$\frac{1}{N} X\left(\frac{2\pi k}{N}\right) = a_k$$

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4. From the given information, we have th Fourier transform $G(e^{j\omega})$ of $g[n]$ to be

$$G(e^{j\omega}) = g[0] + g[1]e^{-j\omega}.$$

Also, when the input to the system is $x[n] = (1/4)^n u[n]$, the output is $g[n]$. Therefore

$$H(e^{j\omega}) = \frac{G(e^{j\omega})}{X(e^{j\omega})}.$$

From Table 5.2, we obtain

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}.$$

Therefore,

$$H(e^{j\omega}) = \{g[0]+g[1]e^{-j\omega}\}\{1-\frac{1}{4}e^{-j\omega}\} = g[0]+\{g[1]-\frac{1}{4}g[0]\}e^{-j\omega}-g[1]e^{-2j\omega}$$

Clearly, $h[n]$ is a three point sequence.
We have

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-2j\omega}$$

and

$$H(e^{j(\omega-\pi)}) = h[0]+h[1]e^{-j(\omega-\pi)}+h[2]e^{-2j(\omega-\pi)} = h[0]-h[1]e^{-j\omega}+h[2]e^{-2j\omega}$$

We see that $H(e^{j\omega}) = H(e^{j(\omega-\pi)})$ only if $h[1] = 0$.
We also have

$$H(e^{j\pi/2}) = h[0] + h[1]e^{-j\pi/2} + h[2]e^{-2j\pi/2} = h[0] - h[2]$$

Since we are also given that $H(e^{j\pi/2}) = 1$, we have

$$h[0] - h[2] = 1.$$

Now note that

$$g[n] = h[n] * \{(1/4)^n u[n]\} = \sum_{k=0}^{2} h[k](1/4)^{n-k}u[n-k]$$

Evaluation this equation at $n = 2$, we have

$$g[2] = 0 = \frac{1}{16}h[0] + \frac{1}{4}h[1] + h[2]$$

Since $h[1] = 0$,

$$\frac{1}{16}h[0] + h[2] = 0.$$

we obtain: $h[0] = \frac{16}{17}$ and $h[2] = -\frac{1}{17}$.
Therefore,

$$h[n] = \frac{16}{17}\delta[n] - \frac{1}{17}\delta[n-2].$$

5.

$$y_1[n] = x[n] - x[n-1] + 3x[n-4], \quad y_2[n] - \frac{1}{3}y_2[n-1] = x[n]$$

(a)

$$y[n] - \frac{1}{3}y[n-1] = x[n] - x[n-1] + 3x[n-4]$$

(b)

$$Y_1(e^{j\omega}) = X_1(e^{j\omega}) - e^{-j\omega}X_1(e^{j\omega}) + 3e^{-4j\omega}X_1(e^{j\omega})$$
$$H_1(e^{j\omega}) = 1 - e^{-j\omega} + 3e^{-4j\omega} \longrightarrow h_1[n] = \delta[n] - \delta[n-1] + 3\delta[n-4]$$

$$Y_2(e^{j\omega}) - \frac{1}{3}e^{-j\omega}Y_2(e^{j\omega}) = X_2(e^{j\omega})$$
$$H_2(e^{j\omega}) = \frac{1}{1-(1/3)e^{-j\omega}} \longrightarrow h_2[n] = \frac{1}{3}^n u[n]$$

$$h[n] = h_1[n] * h_2[n] = (\frac{1}{3})^n u[n] - (\frac{1}{3})^{n-1}u[n-1] + 3(\frac{1}{3})^{n-4}u[n-4]$$

(c)

$$g[n] = h[n] * x[n], \quad f[n] = h[n] * (2x[n-1])$$
$$G(e^{j\omega}) = H(e^{j\omega}).X(e^{j\omega})$$
$$F(e^{j\omega}) = H(e^{j\omega})(2e^{-j\omega}X(e^{j\omega})) = 2e^{-j\omega}(H(e^{j\omega})X(e^{j\omega})) = 2e^{-j\omega}G(e^{j\omega})$$
$$F(e^{j\omega}) = 2e^{-j\omega} \Rightarrow f[n] = 2g[n-1]$$

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6. Reference Book Solution Manual, Page 44-45, Problem 15

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7. (a)

$$x[n] = \delta[n]$$
$$X[k] = \sum_{n=0}^{N-1} \delta[n]W_N^{kn}, \quad 0 \le k \le (N-1)$$
$$X[k] = 1$$

(b)

$$x[n] = \delta[n - n_0], \;\; 0 \leq n_0 \leq N - 1$$

$$X[k] = \sum_{n=0}^{N-1} \delta[n - n_0] W_N^{kn}, \;\; 0 \leq k \leq N - 1$$

$$X[k] = W_N^{kn_0}$$

(c)

$$x[n] = \begin{cases} 1 & n \; even \\ 0 & n \; odd \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \;\; 0 \leq k \leq N - 1$$

$$X[k] = \sum_{n=0}^{(N/2)-1} W_N^{2kn} = \frac{1 - e^{-j2\pi k}}{1 - e^{-j(\pi k/N)}}$$

$$X[k] = \begin{cases} N/2 & k = 0, N/2 \\ 0 & otherwise \end{cases}$$

(d)

$$x[n] = \begin{cases} 1 & 0 \leq n \leq ((N/2) - 1) \\ 0 & N/2 \leq n \leq N - 1 \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \;\; 0 \leq k \leq N - 1$$

$$X[k] = \sum_{n=0}^{(N/2)-1} W_N^{kn} = \frac{1 - e^{-j\pi k}}{1 - e^{-j(2\pi k/N)}}$$

$$X[k] = \begin{cases} N/2 & k = 0 \\ \frac{2}{1 - e^{-j(2\pi k/N)}} & k \; odd \\ 0 & k \; even, \; 0 \leq k \leq N - 1 \end{cases}$$

(e)

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N - 1 \\ 0 & otherwise \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} a^n W_N^{kn}, \;\; 0 \leq k \leq N - 1$$

$$X[k] = \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j(2\pi k)/N}} = \frac{1 - a^N}{1 - ae^{-j(2\pi k)/N}}$$

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8. Given a 20-pt finite-duration sequence $x[n]$:

   (a) We wish to obtain $X(e^{j\omega})|_{\omega=4\pi/5}$ using the smallest DFT possible. A possible size of the DFT is evident by the periodicity of $e^{j\omega}|_{4\pi/5}$. Suppose we choose the size of the DFT to be $M = 5$. The data sequence is 20 points long, so we use the time-aliasing technique. Specifically, we alias $x[n]$ as:

   $$x_1[n] = \sum_{r=-\infty}^{\infty} x[n + 5r]$$

   This aliased version of $x[n]$ is periodic with period 5 now. The 5-pt DFT is computed. The desired value occurs at a frequency corresponding to:

   $$\frac{2\pi k}{N} = \frac{4\pi}{5}$$

   for $N = 5,\ k = 2$, so the desired value may be obtained as $X[k]|_{k=2}$

   (b) Next, we wish to obtain $X(e^{j\omega})|_{\omega=10\pi/27}$.
   The smallest DFT is of size $L = 27$. Since the DFT is larger than the data block size, we pad $x[n]$ with 7 zeros as follows:

   $$x_2[n] = \begin{cases} x[n] & 0 \le n \le 19 \\ 0 & 20 \le n \le 26 \end{cases}$$

   We take the 27-pt DFT, and the desired value corresponds to $X[k]$ evaluated at $k = 5$.
