Signals and Systems - MAE 143A

Practice Final Exam- Winter Quarter 2007

Student name and number ______________________________

For all the questions you need to show ALL your work to get to the answer.

1. (4 points) For the time function

$$f(t) = \int_0^t \tau \sin(2(t-\tau))\, d\tau, \qquad t \geq 0$$

   (i) (2 points) Find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$.

   (ii) (2 points) Using the Initial Value Theorem, find $f(0)$.

Solution:

(i) Observe that $f(t)$ is just a convolution $f(t) = t * \sin 2t$. Therefore its Laplace transform can be computed as follows:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} \cdot \mathcal{L}\{\sin 2t\} = \frac{1}{s^2} \cdot \frac{2}{s+4}\,.$$

(ii) The Initial Value Theorem can be applied for any function $F(s)$, and it says that:

$$f(0) = \lim_{s\to\infty} sF(s)\,.$$

In our case, this leads to:

$$f(0) = \lim_{s\to\infty} \frac{2s}{s^2(s+4)} = 0\,.$$

Note: For the exam you have to know the Laplace transform properties, recognize a convolution, and know how to use the Initial Value Theorem and the Final Value Theorem.

2. (3 points) Given that

$$e^{-4t}1(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s)$$

find the inverse Laplace transforms of:

   (i) $F\left(\frac{s}{3}\right)$,

   (ii) $F(s-2) + F(s+3)$,

   (iii) $\frac{F(s)}{s}$.

Solution: In each of these, we have to use the properties of the Laplace transform.

(i) By the time scaling property,

$$f(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} F\left(\frac{s}{a}\right) .$$

Then,

$$F\left(\frac{s}{3}\right) \quad \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \quad 3f(3t) = 3\mathrm{e}^{-12t}1(3t) = 3\mathrm{e}^{-12t}1(t) .$$

(ii) By the frequency shift property,

$$\mathrm{e}^{-at} f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad F(s+a) .$$

Then,

$$F(s-2) + F(s+3) \quad \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \quad \mathrm{e}^{2t} f(t) + \mathrm{e}^{-3t} f(t) = \mathrm{e}^{-2t}1(t) + \mathrm{e}^{-7t}1(t) .$$

(iii) By the integration property,

$$\int_0^t f(\tau)\, d\tau \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{F(s)}{s} .$$

Then,

$$\frac{F(s)}{s} \quad \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \quad \int_0^t f(\tau)\, d\tau = \int_0^t \mathrm{e}^{-4\tau}\, d\tau = \frac{1}{4}\left(1 - \mathrm{e}^{-4t}\right) .$$

Note: For the exam you need to know how to use the Laplace transform properties (superposition, time shift, time scaling, frequency shift, convolution ...)

3. (4 points) Suppose that a system has a transfer function of the form:

$$P(s) = \frac{s-1}{(s+2)(s+a)} ,$$

for some $a \neq -2$. Knowing that we can expand $P(s)$ in fractions as:

$$P(s) = \frac{A}{s+2} + \frac{3}{s+a} ,$$

(i) find the numerical values of A and a,

(ii) compute the *impulse* response to the system. Is the impulse response signal *stable*?

Solution:

(i) To find the numerical values of A and a we use the partial fraction expansion method corresponding to different poles $p_1 = -2$ and $p_2 = -a$. We have that:

$$(s+2)P(s)\Big|_{s=-2} = A \iff \frac{s-1}{s+a}\Big|_{s=-2} = \frac{-3}{-2+a} = A \qquad (1)$$

and

$$(s+a)P(s)\Big|_{s=-a} = 3 \iff \frac{s-1}{s+2}\Big|_{s=-a} = \frac{-a-1}{-a+2} = 3 \qquad (2)$$

Equations (1) and (2) form a system of independent equations and we can use them to find that $a = \frac{7}{2}$ and $A = -2$.

(ii) The impulse response to the system is by definition $p(t) = \mathcal{L}^{-1}\{P(s)\}$. Using the above partial fraction expansion with the obtained numerical values, we have that:

$$p(t) = \mathcal{L}^{-1}\left\{\frac{-2}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s+\frac{7}{2}}\right\} = -2e^{-2t}1(t) + 3e^{\frac{-7}{2}t}1(t)\,.$$

As we can see, $\lim_{t\to\infty} p(t) = 0$. Since the limit is a finite value, the signal is stable by definition.

Note: You have to know the partial fraction expansion method very well and how to compute standard system responses. You have to know when a signal is stable or unstable.

4. (5 points) Find the *step* response to the system:

$$P(s) = \frac{s+1}{s^2+6s+9}\,.$$

Solution: The step response to $P(s)$ is obtained as $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, where

$$Y(s) = P(s)U(s) \;\; U(s) = \mathcal{L}\{1(t)\} = \frac{1}{s}\,.$$

That is,

$$Y(s) = P(s)\frac{1}{s} = \frac{s+1}{s(s^2+6s+9)}\,.$$

To obtain the inverse Laplace transform of $Y(s)$, we use the partial fraction expansion method. First observe that $s^2+6s+9 = (s+3)^2$. The poles of $Y(s)$ are then $p_1 = -3$ (repeated) and $p_2 = 0$. Since $\deg(s+1) < \deg(s(s+3)^2)$, we can write:

$$Y(s) = \frac{c_1}{s+3} + \frac{c_2}{(s+3)^2} + \frac{c_3}{s}\,.$$

To find the constants c_i, we use Case 2 of the method. In this way,

$$\begin{aligned} c_2 &= (s+3)^2 Y(s)\big|_{s=-3} = \frac{s+1}{s}\bigg|_{s=-3} = \frac{2}{3}\,, \\ c_1 &= \frac{d}{ds}\left((s+3)^2 Y(s)\right)\big|_{s=-3} = \frac{d}{ds}\left(\frac{s+1}{s}\right)\bigg|_{s=-3} = \frac{-1}{s^2}\bigg|_{s=-3} = \frac{-1}{9}\,, \\ c_3 &= sY(s)\big|_{s=0} = \frac{s+1}{(s+3)^2}\bigg|_{s=0} = \frac{1}{9}\,. \end{aligned}$$

From here,

$$Y(s) = \frac{-1}{9(s+3)} + \frac{2}{3(s+3)^2} + \frac{1}{9s}\,,$$

and the step response to the system becomes:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{9}\mathrm{e}^{-3t}1(t) + \frac{2}{3}t\mathrm{e}^{-3t}1(t) + \frac{1}{9}1(t)\,.$$

5. (2 points) For the system with transfer function:

$$P(s) = \frac{s+1}{s^2(s+3)}$$

find the differential equation governing the relationship between inputs $u(t)$ and outputs $y(t)$. (Assume zero initial conditions for $y(t)$ and $u(t)$.)

Solution: We have that:

$$Y(s) = \frac{s+1}{s^2(s+3)}U(s)\,,$$

for any $Y(s) = \mathcal{L}\{y(t)\}$, and $U(s) = \mathcal{L}\{u(t)\}$. This implies:

$$(s^3 + 3s^2)Y(s) = (s+1)U(s) \qquad \Longrightarrow \qquad \dddot{y} + 3\ddot{y} = \dot{u} + u\,.$$

Note: You need to know how to pass from a Time Domain system representation to a Frequency Domain representation and vice versa. This applies for both continuous-time and discrete-time systems.

6. (9 points) A system is described by the ODE:

$$\ddot{x}(t) - \dot{x}(t) = u(t)\,.$$

Using the Laplace transform method, find the solution to the system with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ and driven by the input $u(t) = \sin \pi t$.

Solution: The Laplace transform of the above ODE with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ and the input $u(t)$ is given as:

$$s^2X(s) - s - sX(s) + 1 = \mathcal{L}\{\sin \pi t\} = \frac{\pi}{s^2 + \pi^2} \qquad \Longleftrightarrow$$
$$s(s-1)X(s) - (s-1) = \frac{\pi}{s^2 + \pi^2} .$$

In other words,

$$X(s) = \frac{\pi}{s(s-1)(s^2+\pi^2)} + \frac{1}{s} .$$

The solution is obtained as $x(t) = \mathcal{L}^{-1}\{X(s)\}$. To find its expression, we make use of the partial fraction expansion method. That is, we expand

$$Y(s) = \frac{\pi}{s(s-1)(s^2+\pi^2)} = \frac{c_1}{s} + \frac{c_2}{s-1} + \frac{c_3}{s-j\pi} + \frac{c_4}{s+j\pi} ,$$

which is the first fraction in $X(s)$, and substitute the result in $X(s)$. The roots are all different, so the constants can be computed as follows:

$$c_1 = sY(s)|_{s=0} = \left.\frac{\pi}{(s-1)(s^2+\pi^2)}\right|_{s=0} = \frac{\pi}{(-1)(\pi^2)} = -\frac{1}{\pi} ,$$
$$c_2 = (s-1)Y(s)|_{s=1} = \left.\frac{\pi}{s(s^2+\pi^2)}\right|_{s=1} = \frac{\pi}{1+\pi^2} ,$$
$$c_3 = (s-j\pi)Y(s)|_{s=j\pi} = \left.\frac{\pi}{s(s-1)(s+j\pi)}\right|_{s=j\pi}$$
$$= \frac{1}{2\pi(1-j\pi)} = \frac{1+j\pi}{2\pi(1-j\pi)(1+j\pi)} = \frac{1}{2\pi(1+\pi^2)} + j\frac{1}{2(1+\pi^2)} ,$$
$$c_4 = \bar{c}_3 = \frac{1}{2\pi(1+\pi^2)} - j\frac{1}{2(1+\pi^2)} .$$

From here, we find a partial fraction expansion of $X(s)$ given as:

$$X(s) = \left(\frac{\pi-1}{\pi}\right)\frac{1}{s} + \left(\frac{\pi}{1+\pi^2}\right)\frac{1}{s-1} +$$
$$+ \left(\frac{1}{2\pi(1+\pi^2)} + j\frac{1}{2(1+\pi^2)}\right)\frac{1}{s-j\pi} + \left(\frac{1}{2\pi(1+\pi^2)} - j\frac{1}{2(1+\pi^2)}\right)\frac{1}{s+j\pi} .$$

The inverse Laplace transform of $X(s)$ and the solution to the ODE is then:

$$x(t) = \mathcal{L}^{-1}\{X(s)\} =$$
$$\left(\frac{\pi-1}{\pi}\right)1(t) + \left(\frac{\pi}{1+\pi^2}\right)\mathrm{e}^{t}1(t)$$
$$+ \left(\frac{1}{2\pi(1+\pi^2)} + j\frac{1}{2(1+\pi^2)}\right)\mathrm{e}^{j\pi t}1(t) + \left(\frac{1}{2\pi(1+\pi^2)} - j\frac{1}{2(1+\pi^2)}\right)\mathrm{e}^{-j\pi t}1(t) .$$

The two complex exponentials can be rewritten using a formula given in class:

$$(a+jb)\mathrm{e}^{(c+jd)t}+(a-jb)\mathrm{e}^{(c-jd)t}=|a+jb|\,\mathrm{e}^{ct}\cos\left(dt+\angle(a+jb)\right)\,, \tag{3}$$

where

$$|a+jb|=\sqrt{a^2+b^2}\,,\qquad \angle(a+jb)=\arctan\frac{b}{a}\,.$$

For the particular numbers we have in $x(t)$, formula (3) leads to:

$$\left(\frac{1}{2\pi(1+\pi^2)}+j\frac{1}{2(1+\pi^2)}\right)\mathrm{e}^{j\pi t}+\left(\frac{1}{2\pi(1+\pi^2)}-j\frac{1}{2(1+\pi^2)}\right)\mathrm{e}^{-j\pi t}=$$
$$\frac{1}{2\pi\sqrt{1+\pi^2}}\mathrm{e}^0\cos\left(\pi t+\arctan\pi\right)=\frac{1}{2\pi\sqrt{1+\pi^2}}\cos\left(\pi t+\arctan\pi\right)$$

Summarizing, the expression for the solution $x(t)$ becomes:

$$x(t)=\left(\frac{\pi-1}{\pi}\right)1(t)+\left(\frac{\pi}{1+\pi^2}\right)\mathrm{e}^t1(t)+\frac{1}{2\pi\sqrt{1+\pi^2}}\cos\left(\pi t+\arctan\pi\right)1(t)\,.$$

7. (3 points) A signal $\{y_k\}$ is related to another signal $\{x_k\}$ by means of the formula:

$$y_k=\sum_{m=0}^{k}x_m\,,\qquad k\geq 0\,.$$

If $\mathcal{Z}\{y_k\}=\dfrac{1}{(z-1)^2}$ and $x_0=0$, what are the numerical values of x_1 and x_2?

Solution: Observe that the formula above for y_k is in fact a convolution:

$$y_k=x_k*1_k=\sum_{m=0}^{k}x_m1_{k-m}=\sum_{m=0}^{k}x_m\,,\qquad k\geq 0\,.$$

Therefore, by the properties of the $\mathcal{Z}$ transform, we have that

$$\frac{1}{(z-1)^2}=X(z)\frac{z}{z-1}\qquad\Longrightarrow\qquad X(z)=\frac{1}{z(z-1)}=\frac{-1}{z}+\frac{1}{z-1}\,.$$

To take the inverse $\mathcal{Z}$ transform, we multiply both sides of the last equation by z:

$$zX(z)=-1+\frac{z}{z-1}\,.$$

In this way,

$$\mathcal{Z}^{-1}\{zX(z)\}=x_{k+1}=-\delta_k+1_k\,,\qquad k\geq 0\,.$$

From here,

$$x_1 = -\delta_0 + 1_0 = -1 + 1 = 0\,, \qquad x_2 = -\delta_1 + 1_1 = 0 + 1 = 1\,.$$

Note: You need to have practice with the properties of the $\mathcal{Z}$ transform and know how to obtain the $\mathcal{Z}$ transforms of signals.

8. (10 points) Solve the following difference equation using the $\mathcal{Z}$ transform method:

$$y_{k+2} + 7y_{k+1} + 10y_k = (-1)^k 1_k$$

with initial conditions $y_0 = 1$, $y_1 = -5$.

Solution: Denote by $Y(z) = \mathcal{Z}\{y_k\}$. Consider the general formula given in class:

$$\mathcal{Z}\{y_{k+n}\} = z^n Y(z) - \sum_{l=1}^{n} z^l y_{n-l}\,, \qquad k \text{ is fixed and } n \geq 1\,.$$

Using this formula for $n = 1$ and $n = 2$ and substituting in the values for the initial conditions, we obtain:

$$\mathcal{Z}\{y_{k+1}\} = zY(z) - zy_{1-1} = zY(z) - zy_0 = zY(z) - z\,,$$
$$\mathcal{Z}\{y_{k+2}\} = z^2Y(z) - zy_{2-1} - z^2y_{2-2} = z^2Y(z) - zy_1 - z^2y_0 = z^2Y(z) + 5z - z^2\,.$$

On the other hand, using the tables for the $\mathcal{Z}$ transform,

$$\mathcal{Z}\{(-1)^k 1_k\} = \frac{z}{z+1}\,.$$

In all, the $\mathcal{Z}$ transform of the difference equation becomes:

$$(z^2Y(z) + 5z - z^2) + 7(zY(z) - z) + 10Y(z) = \frac{z}{z+1}\,,$$

or, equivalently,

$$(z^2 + 7z + 10)Y(z) = \frac{z}{z+1} + z^2 + 2z\,.$$

Solving for $Y(z)$ in this equation leads to:

$$Y(z) = \frac{z}{(z+1)(z^2+7z+10)} + \frac{z(z+2)}{z^2+7z+10}\,.$$

Now observe that $z^2 + 7z + 10 = (z+2)(z+5)$, which leads to the simplification:

$$Y(z) = \frac{z}{(z+1)(z+2)(z+5)} + \frac{z}{z+5} = zX(z) + \frac{z}{z+5}\,. \tag{4}$$

To obtain the inverse $\mathcal{Z}$ transform of $Y(z)$ using the tables, we want to expand $Y(z)$ in fractions of the form:

$$Y(z) = \frac{Az}{z+5} + \frac{Bz}{z+1} + \frac{Cz}{z+2}.$$

We can use the partial fraction expansion method to do that. (*Note:* Before doing anything, notice that the partial fraction expansion method gives you expansions of the form:

$$Y(z) = \frac{A}{z+5} + \frac{B}{z+1} + \frac{C}{z+2},$$

since we want z multiplying in the numerators, we will use the partial fraction expansion method for $\frac{Y(z)}{z}$.) To save some computations, we will just expand the term $X(z)$ in the expression for $Y(z)$ (see equation (4)). Then we will substitute the result back in $Y(z)$.

Let $X(z)$ be the fraction:

$$X(z) = \frac{1}{(z+1)(z+2)(z+5)} = \frac{A}{z+5} + \frac{B}{z+1} + \frac{C}{z+2}.$$

Then we compute the constants A, B and C as follows:

$$\begin{aligned}
A &= (z+5)X(z)|_{z=-5} = \left.\frac{1}{(z+1)(z+2)}\right|_{z=-5} = \frac{1}{(-4)(-3)} = \frac{1}{12}, \\
B &= (z+1)X(z)|_{z=-1} = \left.\frac{1}{(z+5)(z+2)}\right|_{z=-1} = \frac{1}{4}, \\
C &= (z+5)X(z)|_{z=-2} = \left.\frac{1}{(z+5)(z+1)}\right|_{z=-2} = \frac{1}{3(-1)} = -\frac{1}{3}.
\end{aligned}$$

Substituting the expansion for $X(z)$ into $\frac{Y(z)}{z}$ we obtain:

$$\frac{Y(z)}{z} = \frac{1}{12(z+5)} + \frac{1}{4(z+1)} - \frac{1}{3(z+2)} + \frac{1}{z+5},$$

which implies that:

$$Y(z) = \left(\frac{1}{12} + 1\right)\frac{z}{z+5} + \frac{z}{4(z+1)} - \frac{z}{3(z+2)}.$$

Using the tables, the inverse $\mathcal{Z}$ transform of $Y(z)$ becomes:

$$y_k = \mathcal{Z}^{-1}\{Y(z)\} = \left(\frac{1}{12} + 1\right)(-5)^k 1_k + \frac{1}{4}(-1)^k 1_k - \frac{1}{3}(-2)^k 1_k.$$

This gives the solution y_k for $k \geq 0$.

Note: To make sure you got the right solution, you can check if this equation satisfies the initial conditions. In fact,

$$1 = y_0 = \left(\frac{1}{12} + 1\right)(-5)^0 + \frac{1}{4}(-1)^0 - \frac{1}{3}(-2)^0 = \frac{1}{12} + 1 + \frac{1}{4} - \frac{1}{3} = 1\,,$$

$$-5 = y_1 = \left(\frac{1}{12} + 1\right)(-5) + \frac{1}{4}(-1) - \frac{1}{3}(-2) = \frac{-5}{12} - 5 - \frac{1}{4} + \frac{2}{3} = -5\,.$$

Therefore the y_k we have obtained satisfies the initial conditions.